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Излази два пута годишње

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For the publisher
Prof. Violeta Jovanović, PhD, University of Kragujevac, Faculty of Education in Jagodina, Serbia

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EDITORIAL NOTES

Scientific journals in Serbia, whose aim and scope is research in mathematics education, is quite scarce. There is only one journal, *Nastava matematike* [*The Teaching of Mathematics*], and its 'sister' journal published in the English language, *The Teaching of Mathematics*, both published by the Mathematical Society of Serbia and both of which are dedicated to teaching and learning mathematics (the former is practice oriented and the latter research oriented). In the last decade there also have been special thematic issues of journals published that are focused on education in general and some of these in particular on mathematics education and the history of mathematics (*Teaching Innovations* 2014, 2019; Lawrence, 2021).

Dedicating a whole issue of a journal to the thematic issue of mathematics education is therefore not only a novel but truly unique approach in the region.

When preparing *Call for papers* for this special issue of *Uzdanica* journal – *Methodology of Teaching Mathematics*, we decided to introduce a broad range of topics. The reason behind that lies in the fact that we wanted to explore what the main and current topics of interest in mathematics education research in Serbia are.

The articles in this special issue of *Uzdanica* journal are focused on different aspects of mathematics education and teaching and learning mathematics. The special issue brings together research across the field of mathematics education, psychology, pedagogy, philosophy, and STEAM education.

This special issue of the journal also follows an effort of some years to establish a research group across the region that looks at the issues and links between the history of mathematics and mathematics education. In 2018, the Faculty of Education in Jagodina, University of Kragujevac, together with the Faculty of Education of the University of Belgrade, was given a grant by the European Society for Research in Mathematics Education (ERME) to organize an international conference and initiate a research group aligned with one of the ERME's research remits/groups. Our small (three original members, the authors of this paper) group was successful in attracting this funding related to the history of mathematics and mathematics education, and we were able to organize the conference in October

2018 in Jagodina. Unfortunately, the pandemic meant there was some interruption to our joint work since then. But in 2021, the new Doctoral programme at the Faculty of Education in Jagodina was initiated, giving an option of a history of mathematics module for the first time in Serbia at the doctoral level. At the same time, we began putting together the programme for this meeting, the publication to mark it, and plans to have a discussion also on the possibility of founding a history of mathematics society for the region to include not only Serbia but also our neighbours.

In putting together the special issue of this journal we (the editorial team) were faced with challenges reconciling the different methodologies, approaches, and valuable outcomes each field uses and produces in their research practice. We attempted to balance reviewers' individual field's expertise as well as their interests to ensure the accepted papers would meet the methodological standards within and across the relevant fields and disciplines, while purposefully showcasing representative approaches that might complement or be adopted by researchers from other fields.

In the end, what has emerged is, we believe, a very interesting and even unique collection of thirteen papers, with contributions that range in terms of authors from early career to experienced and senior researchers from many European countries. We hope that many of the authors of these papers will continue collaborating in the future and support new, emergent research that aims to identify and record the historical scholarship in mathematics and mathematics education for the geographical region.

THE CONTRIBUTIONS

Lawrence et al. give an insight into a project done in a North-London university and make parallels between the marginalization that happens now and that which happened in history to mathematicians from different backgrounds. Learning from historical examples, the authors draw a conclusion that studying how this marginalization happens may be a very useful addition to the mathematics education of the new entrants to the mathematical profession.

Several research papers dealt with methodological issues in the domain of numbers in teaching mathematics in lower grades of elementary school. **Zeljić, Dabić Boričić, and Ilić** investigate the differences in success and strategies for solving relational terms in comparison-combination tasks of students in the second, fourth, and sixth grades of elementary school. Three types of problems were investigated according to the complexity of their language structure. With its results, the work clearly puts us in the position of thinking about instructions in mathematics teaching aimed at understanding mathematical problems and strategies for solving them, as well as expanding students' experiences on different types of tasks that require planning the solution process and the application of mathematical modeling. **Milinković and Simić** investigated the effects of visually presented tasks within the framework of problem-based teaching on the development of the ability to do mathematical modeling and the process of solving equations and inequalities. The descriptive method was used to analyze, process, and interpret the results of the research in order to examine the types of errors made by fourth-grade elementary school students when working with the visually presented information. The authors draw our attention to visually presented mathematical problems when solving simple and complex equations and inequalities, as well as when composing textual problems based on given iconic representations in elementary mathematics classes. **Milinković, Maričić, and Lazić** examined the students' understanding of an equal sign as a sign that expresses equivalence but also pointed out the progress of the students in forming the concept of equality in mathematics lessons, from operational to relational understanding. The progress in understanding the equality sign as a symbol of equivalence at the age of second to fourth grade of elementary school was investigated. The authors draw our attention to the results obtained and the typical mistakes that students make, citing them as an aid in the design of instructions in the teaching process in order to eliminate the problems of misunderstanding the equality sign as a sign of equivalence.

We especially single out work from the geometry domain, in which the authors deal with preschool children and the field of elementary education as well. The paper of **Vorkapić, Milošević, and Đokić** deals with the ability of imaginary perspective-taking in children of preschool and younger school age, with particular regard to certain components that have not had enough research attention until now, not only in Serbia but in the research community of mathematics education in general. The authors single out visibility and appearance as special perspective-taking abilities and research them, measure, and discuss the results of the selected sample.

The comparison of the success of the preschool children in Serbia to the success of the children from Cyprus and the Netherlands from existing research by Van den Heuvel-Panhuizen, Elia, and Robitzsch (2015) is particularly interesting. With this, the authors try to make the mathematics education research community in Serbia pay attention to the ability to take a child's perspective as a particular skill of spatial reasoning, and what should be further nurtured in early education for children to develop geometric ability.

COVID-19 produced a series of research papers on mathematics education, so we find them in our next topic. **Gorjanac Ranitović et al.** investigated teachers' perception of the requirements and benefits of using indirect versus direct instruction in online mathematics teaching. The authors examined the relationship between teachers' perceptions with socio-educational variables: work environment, level of education, and years of work experience. Moreover, they investigated whether, compared to other subjects, teachers more often apply a certain type of instruction in mathematics classes and what teaching materials and tools for communication they use when applying direct and indirect instruction in online mathematics teaching. The results showed that the examined socio-educational factors, levels of education, and work experience proved to be significantly related to the teachers' perception of the application of direct and indirect instruction.

The paper of **Vulović, Mihajlović, and Milikić** examined the achievements of the students of lower grades in elementary school in mathematics competitions during the COVID-19 pandemic in Serbia. The authors investigated whether the changed conditions in which regular and additional classes were performed in the third and fourth grades of primary school influenced the achievements of the best-performing math students by examining the adoption of advanced-level mathematical content. According to the large sample of nearly 4,000 third-grade students and about 3,800 fourth-grade students in primary school, the results are alarming. By looking at the achievements of students of the same generation through two consecutive competition cycles, the authors observed that insufficiently formed concepts in the third grade, in the first year of the pandemic, remained unexplained by the transition to a higher grade, which represents a problem for the further advancement of students and the development of their mathematical abilities.

Teaching practice or teaching practicum represents an integral part of all initial teacher education programmes. It is considered as one of the key aspects of pre-service teachers' training since it is where they have to put theory into practice (Akkoç, Balkanlıoğlu, Yeşildere-İmre 2016; Makamure, Jita 2019). **Slezakova's** paper in this collection interprets pre-service mathematics and science teachers' responses to a survey which investigates how they appraise their teaching practicums. Additionally, she presents suggestions and recommendations for how to effectively improve the organizational system of these practical teacher trainings.

Milenković investigated the impact of using mind maps on the achievement of mathematically gifted students. The author found out that methodological approach which involves creating mind maps has positive impact not just on students' achievement, but also on systematization of knowledge about Algebraic structures.

Living in the ever-changing world requires us to prepare children to be able to respond to various challenges of modern society. This means they have to acquire knowledge and skills to solve problems and think critically and creatively. Making connections across different disciplines through integrative STEAM education contributes to students' functional knowledge and develops creative thinking and scientific inquiry skills by engaging them in real-world problems (Li et al. 2022). When it comes to the learning of mathematics, the STEAM approach provides meaningful contexts and promotes the use of hands-on activities linked to real world problems (Fitzallen 2015). Considering that teachers have a key role in the teaching process and in preparing young people as future members of society (Atjonen 2015), **Cekić-Jovanović and Gajić** examined the attitudes of elementary school teachers about the importance, place, and role of mathematics and modern technology in STEAM education. The authors found out that elementary school teachers who participated in research had positive attitudes and applied modern technology and mathematics in STEAM classes.

Undoubtedly, technology has become an integral part of our life and has had significant influence on the development of all segments of society including education. Furthermore, digital competences are one of eight key competences of lifelong education (The European Parliament and the Council of the European Union 2006). In her report paper, **Milinković** gives an overview of the research related to the use of mobile educational applications in teaching geometry. The significant contribution of this paper is reflected in the author's providing of an example for how mobile educational applications can be evaluated through three important aspects: pedagogical, mathematical, and cognitive.

Incorporating 'arts' into STEAM introduces new competencies and skills, such as divergent thinking, active learning, social, emotional, and interpersonal skills, and cultural competency (Huser et al. 2020). It also provides opportunities for strengthening learning and cognitive development in meaningful and intentional ways (Dell'Erba 2019). In their report paper, **Milić and Mladenović** investigated possibilities to integrate arts and science contents in order to create meaningful contexts for learning some mathematical concepts in kindergarten. The authors gave examples which might be used in educating future kindergarten teachers.

Čulina gives an interesting point of view and tries to give an answer to the question as to what mathematics we should teach preschool children. The author argues that existing educational standards are limiting and narrowly focused on number and geometry contents, while putting other contents to the side. He gives rich practical examples which could be used for developing mathematical concepts in preschool children.

The last chapter of this issue contains two **book reviews**, the book *New Year's present from a mathematician* by Snezana Lawrence and the textbook *Methodology of teaching Mathematics* by Mirko Dejić, Milana Egerić and Aleksandra Mihajlović.

Snežana Lawrence
Aleksandra Mihajlović
Olivera Đokić

REFERENCES

Akkoc, Balkanlioglu, Yesildere-Imrie (2016): H. Akkoc, M. A. Balkanlioglu, S. Yesildere-Imrie, Exploring Preservice Mathematics Teachers' Perception of Mathematics Teacher through Communities of Practice, *Mathematics Teacher Education and Development*, 18(1), 37–51

Atjonen (2015): P. Atjonen, The importance of the teaching profession in Finland in the 21st century, *Edukacja Quarterly*, 2(133), 5–23.

Dell'Erba (2019): M. Dell'Erba, *Preparing Students for Learning, Work and Life Through STEAM Education*, Education Commission of the States. Retrieved in October 2022 from https://files.eric.ed.gov/fulltext/ED598088.pdf.

The European Parliament and the Council of the European Union (2006): Retrieved in October 2022 from: http://data.europa.eu/eli/reco/2006/962/oj.

Fitzallen (2015): N. Fitzallen, STEM Education: What Does Mathematics Have to Offer?, *Mathematics Education Research Group of Australasia*, 237–244.

Huser et al. (2020): J. Huser et al., *STEAM and the Role of the Arts in STEM*, New York: State Education Agency Directors of Arts Education. Retrieved in September 2022 from: https://www.nationalartsstandards.org/sites/default/files/SEADAE-STEAM-WHITEPAPER-2020.pdf.

Li, Luo, Zhao, Zhu, Ma, Liao (2022): J. Li, H. Luo, L. Zhao, M. Zhu, L. Ma, X. Liao, Promoting STEAM Education in Primary School through Cooperative Teaching: A Design-Based Research Study, *Sustainability*, 14, 10333, https://doi.org/10.3390/su141610333.

Lawrence (2021): S. Lawrence, Serbian mathematics education periodicals for the New Millennium, *ZDM Mathematics Education*, 53, 1555–1564.

Makamure, Jita (2019). C. Makamure, L. C. Jita, Teaching practice and pre-service mathematics teachers' teaching knowledge in Zimbabwe: A mixed methods study, *Issues in Educational Research*, 29(3), 858–880, http://www.iier.org.au/iier29/makamure.pdf.

Nastava matematike, http://elib.mi.sanu.ac.rs/pages/browse_publication.php?db=nm.

Teaching Innovations, 2014, 27(3), http://www.inovacijeunastavi.rs/sr/vol-27-no-3/.

Teaching Innovations, 2014, 33(1), http://www.inovacijeunastavi.rs/en/vol-33-no-1en/.

The Teaching of Mathematics, http://www.teaching.math.rs/.

Van den Heuvel-Panhuizen, Elia, Robitzsch (2015): M. Van den Heuvel-Panhuizen, I. Elia, A. Robitzsch, Kindergartners' performance in two types of imaginary perspective-taking, *ZDM Mathematics Education*, 47/3, Germany: Springer Verlag, 345–362.

PAPERS

Snežana Lawrence
Middlesex University, London, UK
Design Engineering and Mathematics
University of Kragujevac
Faculty of Education in Jagodina
(visiting professor)

Alison Megeney

Nick Sharples

Brendan Masterson
Middlesex University, London, UK
Design Engineering and Mathematics

Matthew Jones
University College London, UK
School of Management

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REVISING THE ROLE OF THE HISTORY OF MATHEMATICS IN POST-PANDEMIC WORLD

Abstract: In this short philosophical and discursive paper, the main objective is to reassess a new emergent role of the history of mathematics in order to bring about greater diversity and engagement in the mathematical sciences. The discussion is based around the project undertaken at a North London university and their partner pre-university college, which piloted the larger national project in the UK in the local context. The success of the project, it is further suggested, would greatly benefit from a framework in which the history of mathematics as a humanistic discipline is closely related to viewing mathematics as a virtuous practice. We also include a short summary about the lives and careers of two Serbian mathematicians, Judita Cofman, and Milica Ilić-Dajović, to showcase how learning about the ways in which marginalisation takes place can help students position themselves and contextualise their priorities as they enter the professional mathematics landscape.

Keywords: Levelling up, humanistic mathematics, decolonising mathematics, mathematizing, virtuous practice.

INTRODUCTION

This article relates to the lessons learned post-pandemic in Britain, and more precisely London. We believe that some of its findings can be applied to any setting, including that of the Balkans and Serbia in particular. In this article we will first

s.lawrence@mdx.ac.uk

describe the widening participation project which had very few connections to the history of mathematics. However, the assumptions before the start of the project and the analysis following the conclusion of the project led to the re-examination of the role that the history of mathematics can have in mathematics education and in similar projects in the future (Levelling Up Scheme 2020; Lawrence et al. 2022). This re-examination is directly related to the roles of narratives in mathematics education as we will show.

THE PANDEMIC AND CALLS FOR GREATER DIVERSITY IN EDUCATION

We start our story with how the pandemic made some inequalities in the UK educational system more visible than they had been previously. The lockdowns in UK were of varying durations, but each meant that the majority of the UK's children were not able to attend school in person and hence suffered interruptions in their education (Institute for Government 2021).

By UK law, all children under the age of 18 are required to attend school full-time (or a training or apprenticeship programme) and during the pandemic lockdowns, this had to be organised mainly online. To attend online schooling, children had to have access to computers and the internet. The number of children who could not access online schooling was never fully investigated, but the government established a 'Get help with technology' scheme which allowed such children to get a computer from the government. The children who were eligible to get a computer had to either not have a digital device in their household, have only one such device for the whole household, or have no broadband (access to the internet) at home (Education Statistics Government 2022).

A series of other events also took place during this unhappy period that greatly influenced the sense of inequality of outcomes for different sections of communities. For example, the protests that followed the death of George Floyd in May 2020 reignited attention to the disparity in outcomes between different races and were further articulated in the protests of the Black Lives Matter movement in both the UK and US. The abduction and murder of a young woman on a peaceful street in London by a serving police officer showed the vulnerability of women during the pandemic when the streets were emptied. Both of those events initiated a period of unrest in the UK, particularly concentrated in London (Parliament 2021).

In terms of less turbulent expressions of dissatisfaction, some new phrases were born during the first and most severe pandemic lockdowns in the middle of 2020. These phrases sometimes related to the new virtual world we found ourselves working in and some were to do more with the greater awareness of systemic racism or inequalities in our social institutions (Herrera 2020). The articulation of these phrases showed the significantly growing awareness of the inequality in both the UK and US. This was further followed by an awareness of how the prevalent narratives of the history of mathematics and sciences seem to favour one particular subset of the population: middle aged, middle class, well-to-do, white men. Following this pandemic-induced awakening towards the inequalities seen in everyday life and in historical narratives, some urgent calls were put out to organise conferences and begin the reassessment of history, bring decolonisation to STEM (Science, Technology, Engineering, and Mathematics), and initiate projects to create new resources. All of this was envisaged to give greater support to disadvantaged groups in education and showcase examples and methods of bringing greater diversity to mathematical sciences (Barrow-Green, Stenhouse 2020).

The project that first mentioned *levelling up* was also conjured in the north of the country, financed by a single donor (Levelling up Scheme 2020). It is worth noting that this particular phrase – *levelling up* – was then also used by the UK government beginning in July 2021 (Government 2021) to describe a programme that unveiled additional funding for the disadvantaged communities to 'level up' to those that are not.

THE WIDENING PARTICIPATION PROJECT AND ITS FINDINGS

In view of these events and reports, the founding of the Levelling up Scheme was envisaged as a starting point to inspire A-level students of "under-represented backgrounds in Maths, Physics, and Chemistry" (Levelling up Scheme 2020) to undertake study of these disciplines at the *leading* universities in UK. There are various measures of university ranking, but the one to which this project referred was *The Times' Good University Guide*. According to this guide, the top leading five universities are interchangeably the University of Cambridge, University of Oxford, University of St. Andrews, Imperial College London, and Durham University (the last two swap places sometimes). Their intake shows consistently less than 70% of students to be those who come from state-funded education (HESA 2021). In contrast, students who study at Middlesex University come from predominantly disadvantaged backgrounds; in 2019–2020 we ranked 7th across the Higher Education sector in this respect, and in 2020–2021, we were the 1st (top) in the league of universities in number of students coming from disadvantaged backgrounds studying at a higher education level.

The programme gained recognition from major professional associations in the UK and in Mathematics both from the London Mathematical Society and the Institute of Mathematics and its Applications. The national level project provided funding to these professional bodies to coordinate local engagement, and the universities that took part provided funding for the projects involving their own students and the targeted group of A-level students.

In our university setting, we had a small team who decided to conduct a trial of the project (Lawrence et al. 2022). We do not count as a leading university according to *The Times Good University Guide* but were happy to pilot the project as envisaged by the national team in order to support students with preparation for entering the other universities.

Whilst we conducted the project, we partially used the framework of evaluation of the national project. The national project had overwhelmingly positive reviews, and our pilot similarly showed positive results. However, the question as to whether the targeted group of A-level students had progressed on to undergraduate study in mathematics (or planned to do so) at one of the *leading* universities did not meet our expected outcomes. Whilst the students acquired greater skill in mathematics and therefore increased their ability to gain top grades and pass entrance examination to the targeted group of leading universities, they did not express great interest in this particular aim of the project.

Both the national project and our own project team expected this to be desirable for students. And, despite our project running on a much smaller scale, we discussed this with students and found that the primary impact on A-level students was the increased confidence and improved results they achieved after they had been tutored by our undergraduate mathematics student-teachers. Conversely, the A-level students did not consider support to get to a 'leading' university as one of their priorities.

We found instead that the participants considered various universities for their undergraduate study (not only ours) and that their consideration was not based on the *Good University Guide* annually published by *The Times*. Instead, students looked for (in order of preference) the universities that

a) offered degrees that they would like to study,

b) those that they would be able to get an early employment through (possibly work-based further studies), and

c) those that were easiest for them to access geographically.

Furthermore, we found that our undergraduate students, the student-teachers in the project, and the A-level students from the local college had not lacked confidence in terms of applying to study mathematics, *nor* were they ambiguous as to whether they should study mathematics, in fact they were quite certain of that. They did, however, not find confidence originally (in the project) with which to discuss and work on mathematics together, nor did they find it easy to talk about mathematics with others.

With these findings we began reconsidering two supporting mechanisms trialled before in mathematics education, beginning to form a proposal for another project yet to be undertaken. These supporting mechanisms were the development of narration in the learning of mathematics and the use of the history of mathematics in promoting positive outcomes for students (Lawrence 2008; Lawrence 2016).

THE NARRATIVES IN MATHEMATICS EDUCATION AND THE HISTORY OF MATHEMATICS

The narratives in mathematics education can roughly be divided into those that look at mathematics from the outside and those that deal with mathematics from the viewpoint of a learner. The first type of narration describes mathematics, looks at it as a discipline, and studies the mathematicians as persons within their own contexts. It tends towards an external dialogue within a group or between two or more groups. The latter type, concerning the internal dialogue of a learner, deals with how one develops an internal voice to do mathematics, to reflect on mathematics, to describe to oneself how one copes with doing mathematics, and so on.

These two types of narratives are not completely distinct, and the dialogues with oneself tend to blend towards the narratives of others over time, as one gains confidence to voice them loudly. So, whilst saying that we can divide narratives into two groups, we also mean two groups and everything in between (Lawrence 2016).

I want to add further layers to the understanding of that budding 'internal' dialogue that eventually becomes a confident external dialogue and encourages the positioning of a growing mathematician towards the discipline. This growing confidence and acceptance of mathematics as something that is part of oneself can be seen as a virtuous practice (Aberdein, Rittberg, Tanswell 2021).

This term itself now needs further explanation. In developing one's confidence to deal with mathematics, a learner tests themselves continuously in skill and perceived aptitude (e.g., 'Am I really understanding this?'), but also in seeing themselves further developing into a yet unknown future person who will live life in a certain way. That way is inevitably virtuous in some manner. Very few people can imagine one starting on a mathematical journey thinking they will become hopeless and will never make a penny out of their profession (there is a whole story in here about romanticising unsuccessful artists and possible parallels with unworldly mathematicians, but we will leave that for another paper).

For this process of growing as a person, Fried (2018) offers a humanistic perspective and ascribes an additional role to the history of mathematics. This role is not only that the history of mathematics can contribute to the learner becoming good or even excellent in mathematics, but becoming a fuller person:

> "[…] the history of mathematics, if it is taken seriously, can become a mode of thinking about mathematics and one's own humanness. What I mean by the latter is that by studying the history of mathematics rather than simply using it as a tool – and that means attempting to understand it as a historian does – one becomes aware of how mathematics is something human being do that therefore informs our human

> identity. In this way, the history of mathematics in mathematics education has the potential to make us fuller human beings […]" (Fried 2018: 85)

But how does one connect the internal narrative that allows for some connection between the success of becoming good at mathematics, with, at the same time, a process of becoming a 'fuller human being'? Of all the techniques of studying narratives in mathematics education, the most useful in this respect is to consider 'mathematizing' as a virtuous practice (Kant, Sarikaya 2020).

In this framework, the practice of *mathematizing* is one that is originally put forward by Freudenthal's Realistic Mathematics Education (RME) (Freudenthal 1968) and can be updated to mean making sense of our context and reality through mathematics. As it has been shown earlier (Lawrence 2016: 147), it is a method in the sense of both transcending and re-enacting what one sees others do in order for one to find one's own voice and construct one's own stories about mathematics and their ability in the discipline. Within that method, it is important to *not* use the stories from history that repeat and reinforce the established general narrative of history in which the figures that seemed to be marginal were in fact, marginalised. We will explore two of such figures shortly.

Of course, to make progress with defining how to look at both mathematics as a virtuous practice and the method of developing one's own voice through the history of mathematics, we need to define what mathematizing means. In Freudenthal's model the constant *mathematizing itself* is a virtuous practice. Mathematizing involves all the things that one does to model contexts and problems in mathematical terms, and it can be compared to acts and states such as art appreciation or art production within one's context, in this case applying mathematical tools to appreciating and doing mathematics. Doing various mathematical things including describing, proving, applying, making abstract conclusions, and using mathematical symbols to understand something within a mathematical context, is what Freudenthal calls *mathematizing*:

> "In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself. The result can be a paper, a treatise, a textbook. A systematic textbook is a thing of beauty, a joy for its author, who knows the secret of its architecture and who has the right to be proud of it. […] What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Freudenthal 1968: 7).

Freudental originally, and more recently Kant and Sarikaya, show (Freudental 1991: 10; Kant, Sarikaya 2020: 3410) that the main components of mathematizing are axiomatizing, formalizing, and schematizing.

Building a schema, modelling of a schema, creating schemes to fit complicated data, or eventually representing reality in some mathematical way is closely related to what Lawrence has called the historical landscape of mathematics (Lawrence 2019). This is not an ideal landscape, nor is it a neat or perfect one. Often it is perilous, desert-like, muddy, and untidy, but every so often a vista may show itself to a mathematical student, when the beauty of the mathematical landscape glows in its full glory (Lawrence 2019). Making the learning of mathematics a habitable landscape is akin to making a map in multi-dimensional personal space as one learns things from mathematics and from history of mathematics. It involves learning how to build networks of knowledge and understanding and how to leave some posts with little embarrassing flags to point to where further details should at some point be added to aid understanding.

Through formalizing, focusing on form, using appropriate symbols and formal expression to describe a scheme or an object, whether real or ideal, becomes possible. A young or recent mathematician can start using the tools and methods at their disposal to try to mathematize whatever obstacle they are trying to overcome. As history teaches us, this is possible with a great variety of tools and hence some mathematicians can potentially be viewed (with the hindsight of history) as pragmatists, mystics, sceptics, radicals, or ascetics (Brunson 2020; Lawrence 2019).

Axiomatizing through identifying rules by which we ascribe meaning to compose mappings of various kinds, whilst satisfying the well-known postulates is probably the most delicate of all aspects of becoming mathematician through *mathematizing*. In this particularly intricate and prone-to-mistake process, one needs a friendly guide. Who can offer such an example? The history of mathematics has long been used in this particular way, and the mathematicians of history are thus invoked to offer reassurance to the contemporary students (Lawrence 2008).

Now that we have mapped mathematizing (in Freudenthal's sense) onto the history of mathematics – in Jankvist's (2009) sense of using history as a tool in mathematics education – we can look for examples of how to build a method for de-marginalising groups of mathematicians and making their presence greater in the Mathematical landscape of the future. What possibly can we gain from that? Perhaps the exemplification of principles on which marginalisation takes place and the creation of new tools for young mathematicians to avoid themselves being put in that position.

THE TWO WOMEN FROM SERBIAN HISTORY OF MATHEMATICS

I will give two examples of women who contributed greatly to both Serbian and English (and more widely European) mathematics and mathematics education in the past century.

The first is the mathematician Judita Cofman (1936–2001). Cofman was born in Vršac (in Vojvodina, the northern autonomous province of Serbia), was of Jewish-Hungarian ancestry, and came from a wealthy family who owned one of the largest breweries in the province. Cofman's father was educated in Germany, and she spoke at least three languages from an early age: Serbian, Hungarian, and German. At the end of World War II, the family's wealth was nationalised, and Cofman enrolled to study mathematics at the Faculty of Philosophy at the University of Belgrade, which then had an outpost in Novi Sad in 1954. At the completion of her studies, Cofman became a teacher in Zrenjanin, a town almost exactly halfway between Vršac and Novi Sad, where she stayed for two years. In 1960, the University of Novi Sad became a university in its own right, and Cofman went back to study for her PhD and to become an assistant to the first female professor of mathematics at the university, Mileva Prvanović (1929–2016). When she completed her PhD, Cofman did postdoctoral research as a Humboldt Fellow at Goethe University in Frankfurt am Main, where she stayed from 1964 to 1965. She then moved to work as a lecturer at Imperial College London from 1965 to 1970 and then University of Perugia, Italy in 1970. In 1971, she gained a position at the University of Tübingen in Germany, then moved to the University of Mainz in 1976. But in 1978, Cofman moved back to London and gained employment as a teacher at Putney High School, a private school for girls (Durnova, Lawrence, Beckers 2022; Lawrence 2022).

Cofman was invited to this prestigious private school by the British mathematician Margaret Hyman, neé Crann (1923–1994), who was at the time the Head of Mathematics at the school. Margaret was a well-known mathematics educator and was President of the Mathematical Association of UK from 1974 to 1975. With her husband Walter Hyman (1926–2020), a professor of mathematics from Imperial College (where Cofman worked a decade earlier), Margaret founded the British Olympiad as part of the International Mathematical Olympiad. Another co-founder was their friend and a teacher at Eton College, Norman Routledge (1928–2013). While in England, Cofman published textbooks dedicated to problem solving based on the work she did with talented young mathematicians in mathematics summer camps. She helped organise and run these camps whilst teaching in England. Routledge was later named the most influential teacher of Tim Gowers (1963–) and Stephen Wolfram (1959–), a Fields medallist and professor at the University of Cambridge (Wolfram 2019).

Cofman was also the PhD supervisor of professor Albrecht Beutelspacher (1950–), via whom she left a considerable legacy in mathematics education in Germany (Mathematikum 2022). She did this through her work at the Johannes Gutenberg University Mainz, where Beutelspacher later produced 47 further descendants. In fact, her influence has recently been recognized in Germany through a conference in the winter of 2021 at the Johannes Gutenberg University Mainz that was held in her honour for her contribution to mathematics education in the country.

Judita Cofman is one of those marginalised mathematicians and mathematics educators who did not see in her lifetime the acknowledgement of merit that were due to her (Lawrence 2022; Nikolić 2014).

A second example is the mathematician and mathematics educator Milica Ilić-Dajović. She was born in Paris, where many of the Serbs sought refuge from the advancing Central Powers (Austro-Hungarians, Germans and Bulgarian forces) during WWI. From childhood, Ilić-Dajović spoke French and Serbian and eventually became fluent in German and Russian (Mićić 2019). Ilić completed her mathematics degree at the University of Belgrade where she met her husband Vojin Dajović. From the mid 1950s, Milica became interested in the competitions of secondary school pupils and organised the first such competition for the country in 1958, a year before the first IMO. She then piloted the first similar competition for primary school children (primary school in Yugoslavia covered ages 7–14) in 1965. Milica was the leading educator of her time in the country. She also initiated the formation of the journal *Nastava matematike* (*The Teaching of Mathematics* 1992–2022).

In 1963 to 1964, her husband Vojin had a ten-month stipend to spend at the Mechanical faculty of the Moscow University Lomonosov. Upon his return, Milica translated two textbooks on mathematical problem solving published by the Moscow Mathematical Olympiad team. She edited these texts, one with preparatory questions and one from the Moscow Olympiad in 1966, and wrote introductions to both. Milica became the country representative for the IMOs in the early years after Yugoslavia joined and was the president of the international committee for the IMOs held in Yugoslavia (IX in 1967 and XIX in 1977). She became, with Vojin, the force behind the founding of the Mathematical Gymnasium (or Mathematical Grammar School as it is sometimes called) in Belgrade. Currently, however, her contribution is sometimes interpreted as second to that of her husband. Ilić-Dajović doesn't seem to be a marginal person at all, but rather one that has been at the centre of mathematics education in her time and context; nevertheless, the historical narrative has positioned her in a supportive, marginal role.

What can we learn from her example and that of Cofman's?

CONCLUSION

The somewhat surprising feedback from the pilot project we undertook in the past academic year has been the fact that students didn't share our, or more precisely, the project's assumption that they would aim for one of the *top* universities in the UK. Instead, they prioritised courses and universities that they considered to be best suited to them. A second surprise was the students' openness about how low their confidence to talk about mathematics had been during the project, despite having dedicated themselves to study mathematics and become mathematicians.

By examining these somewhat contradicting aspects of the project in its context (the post-pandemic world), it became clear that some of the negative aspects of the post-pandemic world also matured into positive and real changes within the younger generations about to enter mathematical landscapes. This change we can describe is in expectation rather than in situation. This expectation is related to the choices they make based on their value-systems and priorities. At the same time, these young people are open about talking about problems with taking ownership of *their* choices. They also seem to need some further support to become more able to articulate their position within the mathematical landscape and envision landscapes they want to further develop themselves.

The stories of those such as Cofman or Ilić-Dajović, who were in some way central to their mathematical landscapes and yet became almost invisible within the historical accounts dealing with their time, can perhaps help. Such stories can unearth the mechanisms by which marginalisation happens both within a lifetime (like with Cofman) or afterwards (like with Ilić-Dajović). The central part of their stories are still their mathematics and their professional lives, but it may help to dissect how their contributions were not respected equally as others' based on their gender and social positions. This can be reflected upon with this new post-pandemic world-view to offer some new tools for those who undertake mathematics as a new calling.

REFERENCES

Aberdein, Rittberg, Tanswell (2021): A. Aberdein, C. J. Rittberg, F. S. Tanswell, Virtue theory of mathematical practices: an introduction, *Synthese*, 199(3–4), 10167–10180.

Barrow-Green, Stenhouse (2020): J. Barrow-Green, B. Stenhouse, *Decolonising the Curriculum through the History of Mathematics*. Retrieved in November 2022 from https://www.open.edu/openlearn/science-maths-technology/mathematics-statistics/decolonising-the-curriculum-through-the-history-mathematics.

Brunson (2020): C. Brunson, *Book Review: A New Year's Present from a Mathematician*. Retrieved in November 2022 from https://newbooksnetwork.com/a-new-years-present-from-a-mathematician.

Durnova, Lawrence, Beckers (2022): H. Durnova, S. Lawrence, D. J. Beckers, *Problem solving in mathematics education and the curious case of International Mathematics Olympiad of 1967*, talk presented at the 7th International Conference on the History of Mathematics Education, Mainz, 19–23 September 2022.

Education Statistics Government (2022): *Laptops and Tablets data.* Retrieved in November 2022 from https://explore-education-statistics.service.gov.uk/find-statistics/laptops-and-tablets-data/2022-january.

Freudenthal (1968): H. Freudenthal, Why to teach mathematics so as to be useful?, *Educational Studies in Mathematics*, 1, 3–8.

Freudenthal (1991): H. Freudenthal, *Revisiting mathematics Education, China lectures*, Dordrecht: Kluwer Academic Publishers.

Fried (2018): M. N. Fried, History of Mathematics, Mathematics Education, and the Liberal Arts, In: G. Kaiser et al. (Eds.), *Invited Lectures from the 13th International Congress on Mathematics Education, ICME-13 Monographs*, 85–101.

Government (2021): *The Prime Minister's Levelling-Up Speech*, 15th July 2021. Retrieved in November 2022 from https://www.gov.uk/government/speeches/the-prime-ministers-levelling-up-speech-15-july-2021.

Herrera (2020): T. Herrera, *The 20 Phrases That Defined 2020*, published 18th Dec 2020. Retrieved in November 2022 from https://www.nytimes.com/2020/12/18/style/words-of-the-year-2020.html.

HESA (2021): *Widening participation: UK performance indicators.* Retrieved in November 2022 from https://www.hesa.ac.uk/data-and-analysis/performance-indicators/widening-participation.

Institute for Government (2021): *Timeline of UK government corona virus lockdowns and restrictions*. Retrieved in November 2022 from https://www.instituteforgovernment.org.uk/charts/uk-government-coronavirus-lockdowns.

Jankvist (2009): U. T. Jankvist, A categorization of the 'whys' and 'hows' of using history in mathematics education, *Educational Studies in Mathematics*, 71(3), 235–261, https://doi.org/10.1007/s10649-008-9174-9.

Kant, Sarikaya (2020): D. Kant, D. Sarikaya, Mathematizing as a virtuous practice: different narratives and their consequences for mathematics education and society, *Synthese*, 1999, 3405–3429.

Lawrence (2008): S. Lawrence, Diversity in Mathematics Classroom, *Real histories directory*, Runnymeade Trust.

Lawrence (2016): S. Lawrence, The Old Teacher Euclid and his Science in the Art of Finding One's Mathematical Voice, *Menon: Journal of Educational Research*, 2(1), 146–159.

Lawrence (2019): S. Lawrence, *A New Year's Present from a Mathematician*, Routledge / CRC Press.

Lawrence (2022): S. Lawrence, New Mathematics and the South Slavs – New Math in Yugoslavian context, In: D. De Bock (Ed.), *Modern Mathematics, an international movement*, Springer.

Lawrence, Megeney, Sharples, Masterson, Jones (2022): S. Lawrence, A. Megeney, N. Sharples, B. Masterson, M. Jones, *Widening participation and Levelling up*, Presentation given at the MSOR Network conference (Mathematics, Statistics, Organisational Research), Dundee, 1–2 September 2022.

Levelling up scheme (2020): *Levelling Up STEM*. Retrieved in November 2022 from https://www.levellingupscheme.co.uk.

Mathematikum (2022): *Mathematics Museum Germany*. Retrieved in November 2022 from https://www.mathematikum.de.

Mićić (2019): V. Mićić, Milica Ilić-Dajović, saradnik i urednik časopisa *Nastava matematike*, povodom stogodišnjice rodjenja, *Nastava matematike*, LXIV(1–2).

Nikolić (2014): Aleksandar Nikolić, The work of Judita Cofman on didactics of mathematics, *Teaching Innovations*, 27(3), 105–113.

Parliament (2021): *Protests during the pandemic.* Retrieved in November 2022 from https://publications.parliament.uk/pa/jt5801/jtselect/jtrights/1328/132805.htm.

The Teaching of Mathematics (1992–2022): *Journal of the Serbian Society of Mathematicians*, Belgrade.

Wolfram (2019): S. Wolfram, *A Book from Alan Turing… and a Mysterious Piece of Paper*, Blog, Retrieved in November 2022 from https://writings.stephenwolfram.com/2019/08/a-book-from-alan-turing-and-a-mysterious-piece-of-paper/.

Снежана Лоренс

Универзитет Мидлсекс, Лондон, Уједињено Краљевство
Одсек за инжењерски дизајн и математику
Универзитет у Крагујевцу
Факултет педагошких наука у Јагодини, Србија (гостујући професор)

Елисон Мегни

Ник Шарплс

Брендан Мастерсон

Универзитет Мидлсекс, Лондон, Уједињено Краљевство
Одсек за инжењерски дизајн и математику

Метју Џоунс

Универзитетски колеџ у Лондону, Уједињено Краљевство
Факултет за менаџмент

РЕВИДИРАЊЕ УЛОГЕ ИСТОРИЈЕ МАТЕМАТИКЕ У ПОСТПАНДЕМИЈСКОМ СВЕТУ

Айсшракш: Циљ овог теоријског и прегледног рада јесте преиспитивање нове улоге историје математике у циљу постизања веће различитости и посвећености у области математике. Анализа се заснива на пројекту који је спроведен на Универзитету у Северном Лондону и партнерском колеџу, а који је покренуо један већи национални пројекат у Великој Британији у локалном контексту. За успех пројекта од велике користи би било да се историја математике као хуманистичке дисциплине уско повеже са виђењем математике као добре праксе. У раду је такође дат кратак приказ живота и каријера две српске математичарке, Јудите Кофман и Милице Илић-Дајовић, са циљем да се студенти упознају са видовима маргинализације како би научили да се изборе за своје место и да поставе себи приоритете у тренутку када почињу професионалну каријеру у области математике.

Кључне речи: равноправност, хуманистичка математика, промена улоге математике, математизирање, добра пракса.

Marijana Ž. Zeljić
Milana M. Dabić Boričić
Svetlana M. Ilić
University in Belgrade
Teacher Education Faculty
Department of Didactics of Mathematics

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THE UNDERSTANDING OF RELATIONAL TERMS IN COMPARE-COMBINE WORD PROBLEMS ON DIFFERENT LEVELS OF EDUCATION[1]

Abstract: Routine word problems are thoroughly described and categorized to *combine*, *change,* and *compare* problems. This paper investigates how 2nd, 4th, and 6th-grade students solve integrated combine and compare problems. We used the integrated combine and compare problems with consistent language (CL) formulation, inconsistent language (IL) formulation, or more complex structure. Our research sample consists of 44 students in 2nd grade, 48 students in 4th grade, and 42 students in 6th grade from schools in Belgrade. The results show that students are more successful in solving problems with CL than with IL formulation at all levels of education. Students from the 2nd, 4th, and 6th grade are equally successful in solving the CL problem. The surprising result is the nonexistence of a significant difference in the achievement of students in 4th and 6th grade on the IL problem, which could indicate an obstacle in the development of relational term understanding after introducing algebra into mathematical education. Low achievement on the problem with more complex structure showed that students have issues with the modeling process and that they are not eager to use algebraic strategies or graphical representations. These results imply a need for a systematic approach to teaching routine problems after introducing algebra in mathematics education.

Keywords: word problems, combine problems, compare problems, problem solving strategies, mathematical education.

INTRODUCTION

There are numerous reasons why word problems have been at the center of research in mathematics education for the past few decades. One of them is in their twofold use: they could be used as *routine problems* to facilitate the devel-

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marijana.zeljic@uf.bg.ac.rs

opment of the conceptual knowledge of basic arithmetic operations (Carpenter 1986; Carpenter, Hiebert, Moser 1981; Schroeder, Lester 1989), or they could be used as *non-routine* problems in the way that facilitates mathematical thinking and mathematical literacy in general (Verschaffel et al. 2010; Van Dooren et al. 2010). The categorization of routine problems with one operation to *combine*, *change*, and *compare* problems is broadly described in the literature, and every type of problem is separately investigated (Cummins et al. 1988; De Corte, Verschaffel, De Win 1985; Morales, Shute, Pellegrino 1985; Riley, Greeno, Heller 1983). Research confirmed that compare problems are the most difficult for students (e.g., Carpenter, Moser 1984; Cummins et al. 1988; Nesher, Greeno, Riley 1982; Stern 1993). In addition, compare problems are analyzed by the aspect of their language consistency. A compare problem with consistent language (CL) formulation is less challenging to the students than a compare problem with inconsistent language (IL) formulation. Even though *combine* problems are less complicated for the students than change and compare problems, there are also different types of combine problems that could be less or more challenging to the students (Riley, Greeno 1988).

It is still uninvestigated how students at different education levels solve problems created by the integration of the combine and compare problems. As we have stated, both types of problems are routine, but there are two issues that integration could bring into problem solving: 1) language consistency of the compare problem; 2) more complex structure of the combine problem. In this paper, which is part of broader research, we investigate the achievement and strategies of the students at different levels of education (2nd, 4th and 6th grade) on the integrated combine and compare problems (here referred to as *combine-compare* problems). Students at these levels of education have diverse mathematical skills. It is essential to understand obstacles that students at each level of education have in *combine-compare* problem solving and give implications for their overcoming. In their future mathematics education, solving these problems would become just a tool for solving more complex routine and non-routine problems. Hence, it is important that students can solve them correctly and efficiently.

THE CATEGORIZATION OF WORD PROBLEMS WITH ONE OPERATION

One of the most used definitions is that word problems are verbal descriptions of a problem situation in which the answer could be given by performing mathematical operations on numerical data provided in the text of the problem (Verschaffel, Depaepe, Van Dooren 2014). The above-mentioned routine word problems that could be solved with one mathematical operation (in one step) were key components in mathematics curriculums for elementary schools worldwide. They are considered the basis for learning in mathematics classrooms; hence the voluminous research in the 80s and 90s was focused on problems with one operation and their characteristics, categorization, process of solving, and impact on students' thinking.

The basic categorization of word problems with one operation (addition or subtraction) appeared at the beginning of the 80s. Numerous empirical studies with children aged five to eight showed that even if problems could be solved with the same arithmetic operation, they belong to different semantic categories, which suggests that different strategies for representing and solving problems trigger different types of mistakes (Fuson 1992). Based on the semantic structure and situation described in the text of the problem, they are classified into three categories: *combine*, *change*, and *compare* problems (Cummins et al. 1988; De Corte, Verschaffel, De Win 1985; Morales, Shute, Pellegrino 1985; Powell et al. 2009; Riley, Greeno, Heller 1983; Verschaffel 1994). The categorization served as a guideline for numerous future research studies. The other aspect of categorizing word problems is whether they describe static or dynamic situations (Carpenter, Hiebert, Moser 1981). We provide the examples by De Corte and Verschaffel (1986) that explain the difference between the three categories and the relationships (dynamic/static) in Table 1.

Table 1. Categorization of word problems – semantic structure and dynamic of the situation.

Type	Example	Situation
Change	Pete had 3 apples. Ann gave him 5 more apples. How many apples does Pete have now?	Dynamic situation – implied action in which one set is joined to another; Two entities are the subset of the third.
Combine	Pete has 3 apples. Ann has 5 apples. How many apples do they have altogether?	Static relationship; Two entities are the subset of the third.
Compare	Pete has 3 apples. Ann has 8 apples. How many apples does Ann have more than Pete?	Static relationship; One of the sets described in the problem is completely disjoint from the other two

Riley and Greeno (1988) investigated the achievement of students of different ages in solving problems in all three categories. Students were more successful in the combine word problems than in change word problems and least successful in the compare word problems. Many studies also confirm that compare problems are the biggest challenge for students (Briars, Larkin 1984; Carpenter, Moser 1984; Cummins et al. 1988; Morales, Shute, Pellegrino 1985; Nesher, Greeno, Riley 1982; Okamoto 1996; Riley, Greeno 1988; Riley, Greeno, Heller 1983; Stern 1993).

Research also deals with further analysis and categorization of *change*, *combine,* and *compare* problems (Riley, Greeno 1988).

The *change* problems are subcategorized according to whether the result, change, or start value is unknown. We will not represent the classification of change problems because they are not used in our research.

For the *combine* problems, classification is made according to the position of the unknown entity set. There are two types of these problems: problems with the unknown combination (the total number or the whole) and problems with an unknown subset (part). Some of the examples provided by Riley and Greeno (1988) that have linguistical forms like the one we used in this research are presented in Table 2.

Table 2. Categorization of combine and compare problems.

Category	Subcategory	Example
Combine problems	Combination (the total number or the whole) unknown	(1) Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether?
	Subset (part) unknown	(2) Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have?
Compare problems	Difference unknown	(3) Joe has 5 marbles. Tom has 8 marbles. How many marbles does Tom have more than Joe?
	Compared quantity unknown	(4) Joe has 3 marbles. Tom has 5 more marbles than Joe. How many marbles does Tom have?
	Referent unknown	(5) Joe has 8 marbles. He has 5 more marbles than Tom. How many marbles does Tom have?

There are three types of *compare* problems: when the difference set is unknown, when the compared set is unknown, and when the referent set is unknown (Table 2). In other words, in *combine* problems, any of the entities (the difference, the compared quantity, or the referent) can be left to the students to find. Students are most frequently asked to find the unknown difference. Even if all three types of problems represent the same relationship, the most difficult for the students are the ones with unknown referents; the problems with unknown compared quantities and the problems with unknown differences seem to be the least difficult for the students (Schumacher, Fuchs 2012). One of the reasons why problems with an unknown referent are the most difficult type of compare problem is that they require an understanding of the symmetrical relationship between relations *more than* and *less than* (Stern 1993).

Another approach to classifying compare word problems is based on language formulations. Lewis and Mayer (1987) describe two types of problems: consistent language problems (CL) and inconsistent language problems (IL). In CL problems, the mathematical operation can be easily discovered using the relational term (key term, keyword). For example, if the relational term is *more than*, the task's solution is adding quantities. In contrast, in IL problems, the mathematical operation could not be found by keyword. For example, a problem contains *more than*, but it must be solved by subtraction. In Table 2, problem (4) is a CL problem and problem (5) is an IL problem. We can see that compare problems with an unknown compared quantity have CL formulation and compare problems with a referent unknown have IL formulation.

OBSTACLES IN SOLVING COMPARE WORD PROBLEMS

There are several hypotheses about the source of the difficulties in solving compare problems. Many researchers (Schumacher, Fuchs 2012; Riley, Greeno, Heller 1983; Riley, Greeno 1988; Resnick 1983; Okamoto 1996; Okamoto, Case 1996) emphasize that younger students could not understand that the difference between the number of elements of two sets could be expressed in parallel ways by using terms *more* and *less.* Younger students lack knowledge and experience with language describing quantities' relations. Hence, there is research that implies that students have to learn about the symmetry of the comparison – that sentences "Monica has 11 goats less than Martin", and "Martin has 11 goats more than Monica" can be used to describe the same situation (Okamoto 1996; Okamoto, Case 1996).

Second, the semantical relations between known and unknown quantities could be less or more explicit, which could bring difficulties in understanding the situation described in the problem (De Corte, Verschaffel, De Win 1985; De Corte, Verschaffel, Pauwels 1990; Marzocchi et al. 2002; Verschaffel, De Corte, Pauwels 1992). For successful problem solving, it is essential to understand the situation and these semantical relations (Cummins 1991; Cummins et al. 1988; Kintsch 1988; Kintsch, Greeno 1985).

The third and the most researched hypothesis about the difficulties in solving compare problems is in the consistency of the relational term (i.e., keyword – *more than* or *less than*) used in the problem and the mathematical operation needed for its solving. There are two approaches to solving compare problems (Hegarty, Mayer, Monk 1995). In the first approach, students automatically translate *more than* into addition and *less than* into subtraction and develop a solving plan that implies combining the numbers given in the problem and translated operations. This approach, which is a superficial problem-solving strategy, is related to unsuccessful problem solvers. Researchers refer to this approach differently, as: "compute first and think later" (Stigler, Lee, Stevenson 1990: 15), keyword method (Briars, Larkin 1984), and number grabbing (Littlefield, Rieser 1993).

Riley, Greeno, and Heller (1983) state that students intuitively rely on the automatically activated rule – *add if the relation is 'more' and subtract if the relation is 'less'*. In some word problems, this approach really leads to the correct solution; the numbers and keywords from the text can be translated directly into mathematical expressions, but with these problems, students only practice computing skills and imitate the problem-solving process without using conceptual understanding and logical thinking (Lithner 2008; Boesen et al. 2014). In other words, students using this approach do not construct an adequate situational and mathematical model of a problem.

The other approach, related to successful problem solvers, requires constructing a situational model and using an adequate strategy for its solving. Accordingly, many studies started investigating language consistency in the compare problems.

As we previously mentioned, Lewis and Mayer (1987) recognized two parallel types of problems: CL and IL problems. They confirmed that students generally make more mistakes on IL problems than on CL problems, especially when they choose the mathematical operation. In IL problems, they tend to choose the opposite operation. This is called the consistency effect which is investigated and confirmed in many studies (Hegarty, Mayer, Green 1992; Hegarty, Mayer, Monk 1995; Stern 1993; Verschaffel 1994; Verschaffel, De Corte, Pauwels 1992; Lewis, Mayer 1987; Hegarty, Mayer, Green 1992; Lewis 1989; Verschaffel, De Corte, Pauwels 1992; Pape 2003; Van der Schoot et al. 2009).

We can question and investigate if the consistency effect is related to students' age. Stern (1993) conducted two studies in which he investigated the students' understanding of the symmetry of terms *more* and *less* in solving compare problems with the unknown referent. In these studies, he presented pictures to first graders and asked the students to pair them with relational sentences. For example, students had to state which of the sentences were correct: "there are 2 more cows than pigs", and "there are 2 pigs less than cows". Even if students understood the meaning of the sentences, they did not understand that both relations (more and less) can be used to express the same relationship. Studies also showed that low student achievement on this task was related to their ability to solve compare problems with the unknown referent. As the studies show, one of the possible reasons for difficulties in compare problems is students' incomprehension of relational terminology. Elementary school students do not have the conceptual knowledge needed for a complete understanding of compare problems, which could explain their difficulties in solving this type of problem (Cummins et al. 1988; Riley, Greeno, Heller 1983). They do not have the ability to understand and process the meaning of the problem and recall the adequate problem structure (Koedinger, Nathan 2004).

The findings of Boonen and Jolles (2015) were different. They researched why second graders had more difficulties with compare problems than with combine and change problems. As was expected, students made more mistakes on compare problems than on the other two types, but surprisingly they did not confirm the consistency effect. The second graders in this study were equally successful in solving CL and IL problems. The explanation they provided for these results is that students generally showed difficulties processing the relational terms *more* *than* and *less than*, which could be the reason for the results not confirming the consistency effect.

Later research investigated the consistency effect on students in higher grade levels and university (Pape 2003; Van der Schoot et al. 2009). The effect is confirmed with university students (e. g. Hegarty, Mayer, Monk 1995; Lewis 1989; Lewis, Mayer 1987), higher grade level students (Van der Schoot et al. 2009), and lower-level grade students (Boonen, Jolles 2015; Mwangi, Sweller 1998; Schumacher, Fuchs 2012; Willis, Fuson 1988). These results raise a question as to whether younger students' difficulties with solving compare problems are caused by the formulations of IL problems or by a general lack of understanding of relations in both IL and CL problems.

Research that is somewhat more recent (Nesher, Hershovic, Novotna 2003) investigates compare problems with higher complexity. These problems include comparing three quantities instead of two and relations between them. As we previously stated, the difficulties with simple problems (with two entities) occur because of 1) language consistency, 2) lack of understanding of the symmetry of the operations, or 3) the different structure when the *referent* or the *compared* value is unknown. The difficulties are even greater on problems with higher complexity because there are two comparisons in a single problem. The results of research by Nesher et al. (2003) imply that students' achievement depends on the structure of each problem. It is not emphasized, but the examples used in this study integrated combine and compare problems, to which we refer as combine-compare problems. This integration enables the creation of numerous problems with the different structures. Similarly, when integrating combine and compare problems with two quantities (compared quantity and referent quantity), we can create problems with simple or complex structure.

METHODOLOGY

The study presented in this paper is a part of more extensive research that investigates the students' achievement on and strategies for combine-compare problems. The aim of the study is to investigate if the students' understanding of relational terminology (terms "more than" and "less than") develops with the students' age and if the development is accompanied by greater success in solving problems with more complex structures. For this purpose, we analyzed the subcategories of combine and compare word problems and made three integrations: 1) problem with CL formulation; 2) problem with IL formulation; 3) problem with complex structure. Specifically:

Problem A. CL formulation – CL compare problem and combine problem with an unknown combination (total) number:

Joca has 32 marbles, and David has 20 marbles more than him. How many do they have together?
Subproblem 1: Compare word problem with CL formulation
Finding the number of David's marbles
Subproblem 2: Combine word problem with an unknown combination
Finding the total number of marbles

Problem B. IL formulation – IL compare problem and combine problem with an unknown combination (total) number:

Zoka has 32 marbles, which is 20 marbles less than Angela. How many marbles do they have together?
Subproblem 1: Compare word problem with IL structure
Finding the number of Angela's marbles
Subproblem 2: Combine word problem with an unknown combination
Finding the total number of marbles

Problem C. Complex structure – compare problem and combine problem with an unknown subset:

Zoka and David have 84 marbles in total. David has 20 marbles more than Zoka. How many marbles does each child have?
Subproblem 1: Combine word problem with an unknown subset
Subproblem 2: Compare word problem

Problems A and B have a simple structure, and the difference between them is in the consistency of the language. Problem A is a CL problem that can be solved with the keyword method (Briars, Larkin 1984; Hegarty, Mayer, Monk 1995; Littlefield, Rieser 1993; Riley, Greeno, Heller 1983; Stigler, Lee, Stevenson 1990). Problem B is an IL problem whose solution implies knowing the symmetry of language and operations (Stern 1993). On the other side, Problem C has a more complex structure. It contains a combine problem with an unknown subset. The solution to this problem requires using more sophisticated strategies for solving. The language consistency is irrelevant in this integration.

The sample for our research consisted of 2nd, 4th, and 6th-grade students. Students at this age can use different problem-solving strategies: the 2nd graders are familiar with arithmetic strategies of solving; 4th graders can use basic algebraic notation and a small number of solving strategies, while 6th graders can use algebraic strategies for solving.

We operationalized the aim through the following research questions:

1. What is the achievement of students in certain grades (in 2nd, 4th and 6th grade) on CL and IL problems, and are there differences in the achievement on CL versus IL problems?

2. Is students' achievement on IL and CL problems related to the students' level of mathematics education (i.e., the grade students attend)?

3. Is students' achievement on the problem with the more complex structure related to the students' level of mathematics education (i.e., the grade students attend)?

4. What are students' strategies for solving combine-compare problems, and what are the most common mistakes they make?

Based on the results of previous research directed at students' achievement on combine and compare problems, we formulated the following hypotheses:

1. Students will have higher achievement on CL problems than on IL problems at all levels of education.

2. Students' achievement in solving IL and CL problems will be related to the level of students' education, especially on the problem with a more complex structure.

3. Students' achievement in solving the problem with a more complex structure will be related to the level of students' education, especially on the problem with a more complex structure.

4. Sixth-grade students will use algebraic strategies when solving problems with more complex structures, while younger students will use arithmetic strategies. We expect that the most common mistake will be using the keyword approach on IL problems and that younger students will use it more frequently.

The research sample consists of 134 students from one primary school in Belgrade that cooperates with the researchers' institution. Students are from two classes of 2nd grade (44 students), two classes of 4th grade (48 students), and two classes of 6th grade (42 students). They did not have a time limit to solve problems A, B, and C.

We used the Chi-square independence and homogeneity test to analyze the relationships between variables and differences in achievement. To express the strength of the association, we used Cramer's V coefficient.

RESULTS

The achievement of students on the CL problem (problem A) and IL problem (problem B) is presented in the Table 3.

Table 3. The students' achievement on CL and IL problems

Grade	Correct		Incorrect		Missing		Total
Consistency	CL	IL	CL	IL	CL	IL	
2	36	16	8	28	0	0	44
	81.8%	36.4%	18.2%	63.6%	0.0%	0.0%	100.0%
4	43	33	5	15	0	0	48
	89.6%	68.8%	10.4%	31.3%	0.0%	0.0%	100.0%
6	36	28	4	13	2	1	42
	85.7%	66.7%	9.5%	31.0%	4.8%	2.4%	100.0%
Total	115	77	17	56	2	1	134
	85.8%	57.5%	12.7%	41.8%	1.5%	0.7%	100.0%

We used the Chi-square homogeneity test to examine if there is a difference in the achievement on the CL versus the IL problem. The results of the test presented in Table 4 showed that students' achievement was significantly better on the CL than on the IL problem in 2nd grade (p = .000), in 4th grade (p = .012), and in 6th grade (p = .016).

Table 4. The results of the Chi-square homogeneity test in investigating the difference on CL versus IL problem

Grade	n	df	Chi square (n, df)	p
2	88	1	18.803	.000
4	96	1	6.316	.012
6	81	1	5.753	.016

Furthermore, we applied the Chi-square test of independence to examine the relationship between students' achievement on CL (and IL) problems and level of education. For the CL problem, the test did not show the existence of a significant relationship χ^2 (134, 2) = 1.658, p = 0.437. In Table 3, we can see that across the whole sample (134 students), about 85% of students solved CL problems correctly.

For the IL problem, the Chi-square test of independence showed a statistically significant relationship between the students' achievement and level of education χ^2 (133, 2) = 12.507, p = 0.002, with moderate strength of association r = .307 (Cramer's V coefficient). Further analysis of students' achievement (Table 5) showed that there is no significant difference in students' achievement in 4th and 6th grade (p = .963), but that there are differences between the students' achievement in 2nd grade versus 4th grade (p = .003) and 2nd grade versus 6th grade (p = .002). The results shown in the achievement table (Table 3) imply that about one-third of 2nd graders solved the IL problem correctly, and about two thirds of 4th graders and 6th graders solved this problem correctly.

Table 5. The Chi-square test results when examining the relationship between success and the age of students by grade pairs on a task with inconsistent wording (IL).

Comparison beetwen grades	Chi-square	p	r
2nd and 4th	χ^2 (92, 1) = 9.673	0.003	0.319
2nd and 6th	χ^2 (85, 1) = 8.665	0.002	0.324
4th and 6th	χ^2 (89, 1) = 0.002	0.963	

We have also analyzed the incorrect responses on CL and IL problems. The 2nd graders produced a greater number of errors on CL (18.2%, Table 3), and on IL (63.6%, Table 3) problems than 4th and 6th graders (who produced about 10% on CL and about 31% on IL problems, Table 3). In Table 6 we are presenting the analysis of incorrect responses on CL and IL problems. Students who did not solve the CL problem correctly mainly just added numbers from the text of the problem (11.4% of 2nd graders and 9.5% of 6th graders, while this number was smaller in 4th grade – 4.2%).

On the IL problem, 2nd graders also gave the biggest number of incorrect responses, but this time the error was in the relation term (50%, Table 6). The number of relation term errors was smaller in the 4th and 6th grades (31.3% and 26.2%).

Table 6. The categorization of students' incorrect responses on CL and IL problem

Grade	Uncategorized		Just add numbers		Relation term error		Total	
	CL	IL	CL	IL	CL	IL	CL	IL
2	3 6.8%	4 9.1%	5 11.4%	2 4.5%	/	22 50.0%	8 18.2%	28 63.6%
4	3 6.2%	0 0.0%	2 4.2%	0 0.0%	/	15 31.3%	5 10.4%	15 31.3%
6	0 0.0%	0 0.0%	4 9.5%	2 4.8%	/	11 26.2%	4 9.5%	13 31.0%

The students' achievement on the problem with complex structure is given in Table 7.

Table 7. Student achievement on a task with a more complex structure

Grade	Correct	Incorrect	Missing	Total
2	4 9.1%	34 77.3%	6 13.6%	44 100.0%
4	14 29.2%	21 43.8%	13 27.1%	48 100.0%
6	25 59.5%	15 35.7%	2 4.8%	42 100.0%
Total	21 15.7%	43 32.1%	70 52.2%	134 100.0%

The Chi-square test showed a significant relationship between students' achievement and age χ^2 (134.2) = 32.662, p = 0.000, with moderate strength of association r = .349 (Cramer's V coefficient). Further analysis showed that there are differences between every pair of different grades (Table 8). From Table 7 that shows students' achievements, it can be seen that almost 10% of 2nd graders, almost 30% of 4th graders, and almost 60% of 6th graders solved this problem correctly.

Table 8. The Chi-square test results when examining the relationship between success and age of students by grade pairs on a task with a complex structure

Comparison beetwen grades	Chi-square	p	r
2nd and 4th	χ^2 (92, 2) = 11.054	0.004	0.347
2nd and 6th	χ^2 (86, 2) = 24.541	0.000	0.534
4th and 6th	χ^2 (90, 2) = 11.822	0.003	0.362

Analysis of incorrect responses implies that students used superficial strategies. Second-grade students mainly used superficial strategies (almost 40%), while 4th and 6th-grade students made this mistake in roughly 8% and 7% of responses, respectively (Table 9).

Table 9. Students' mistakes when solving a task with a complex structure

Grade	Uncategorized	Superficial strategy	Error in relational term	Total
2	17 38.6%	17 38.6%	0 0.0%	34 77.3%
4	16 33.3%	4 8.3%	1 2.1%	21 43.8%
6	12 28.6%	3 7.1%	0 0.0%	15 35.7%

Both problems, with CL and IL formulation, could be solved using arithmetic strategy by performing arithmetical operations on numbers provided in the text of the problem. A task with a more complex structure allowed students to use different arithmetic and algebraic strategies. However, the algebraic strategy was used in a small number of cases: 2 (4.1%) 4th-grade students used an algebraic strategy, and 3 (7.1%) 6th-grade students (2 of whom only wrote the relations with symbols, then continued with the arithmetic strategy). However, we recognized different arithmetic strategies used by students:

1. Start from equal sets strategy, in which students start from the equal sets and make a difference between (e.g. 84 : 2 ± 10);
2. Start from the difference between sets strategy;
3. Guessing the quantities based on the solution of CL and IL problems.

The frequency and percentage of each strategy are provided in Table 10.

Table 10. Arithmetic strategies of students when solving a task with a complex structure

Grade	Start from equal sets	Start from the difference between sets	Guessing the quantities
2	1 2.3%	2 4.5%	0 0.0%
4	8 16.7%	7 14.6%	5 10.4%
6	11 26.2%	13 31.0%	7 16.7%

In addition to choosing a strategy, we were also interested in which strategies lead to the correct solution in most cases. Table 11 shows that the strategy that starts from the difference between the sets leads to the correct solution in more cases: 73% of the students who used this strategy solved the problem correctly, against 45% of the students who used the start from equal sets strategy.

Table 11. Choice of strategy and accuracy of completed tasks with a more complex structure on the entire sample (2nd, 4th and 6th-grade students)

	Start from equal sets	Start from the difference between sets
Correct	9 45.0%	16 72.7%
Incorrect	11 55.0%	6 27.3%
Total	20 100.0%	22 100.0%

DISCUSSION

Even though combine and compare problems and their integration are widespread in primary school mathematics, two aspects of integration need to be illuminated. First is the aspect of language consistency that compare problem brings to the integration, and the second is the problem of a more complex structure for cases in which the combine problem has an unknown subset.

Our first two research questions refer to language consistency – the achievement on CL and IL problems at different levels of education (2nd, 4th, and 6th grade) and the relationship between achievement and the levels. As we posed in the theoretical part of the paper, previous research in compare problem solving is mostly focused on the understanding of relational terms (Schumacher, Fuchs 2012; Riley, Greeno, Heller 1983; Riley, Greeno, 1988; Resnick 1983; Okamoto

1996; Okamoto, Case 1996; Cummins et al. 1988; Riley, Greeno, Heller 1983; Stern 1993) with emphasis on language consistency (Hegarty, Mayer, Green 1992; Hegarty, Mayer, Monk 1995; Stern 1993; Verschaffel 1994; Verschaffel, De Corte, Pauwels 1992; Lewis. Mayer 1987; Hegarty, Mayer, Green 1992; Lewis 1989; Verschaffel, De Corte, Pauwels 1992; Pape 2003; Van der Schoot et al. 2009). Our results showed that students have significantly higher achievement on the CL problem (Problem A) than on the IL problem (Problem B) in each level of education we investigated. This result is in accordance with previous research, which reports the consistency effect on different levels of education (Schumacher, Fuchs 2012; Riley, Greeno, Heller 1983; Riley, Greeno 1988; Resnick 1983; Okamoto 1996; Okamoto, Case 1996; Pape 2003; Van der Schoot et al. 2009; Hegarty, Mayer, Monk 1995; Lewis 1989; Lewis, Mayer 1987; Schumacher, Fuchs 2012; Willis, Fuson 1988).

Our results also showed no significant relationship between students' achievement on the CL problem and students' level of education. Regarding this result, it raises concern that about 15% of students in all grades did not solve the CL problem correctly (Table 3). These students do not have the conceptual knowledge required to solve this problem (Cummins et al. 1988; Riley, Greeno, Heller 1983). All the other problems that are more complex than the CL problem will stay out of their reach, which could cause difficulties in their future mathematics education.

The results of the analysis of the IL problem show significant differences between students' achievement and level of education. The consistency effect is the strongest in the 2nd grade (36% solved IL problem correctly, Table 3) but fades in the 4th grade (69% solved IL correctly, Table 3) and in the 6th grade (67% solved IL problem correctly, Table 3). Besides, the analysis showed no significant difference between 4th and 6th graders' achievement on the IL problem (Table 5). Surprisingly, two years of teaching algebra and arithmetic did not influence the level of understanding of relations between quantities.

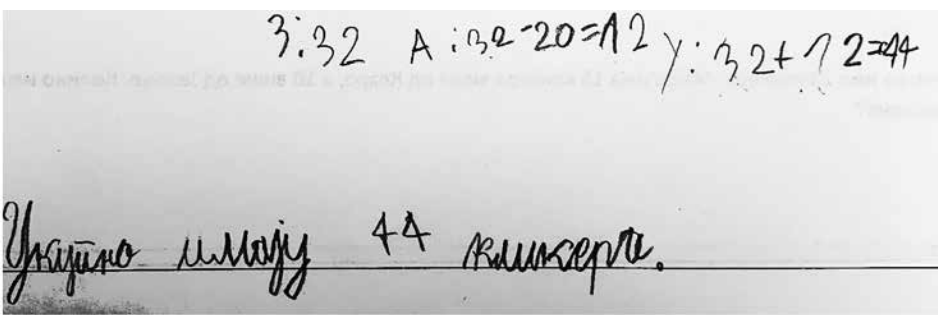
We found two possible guidelines in the literature for improving achievement. First, Boonen and Jolles (2015) showed that instruction focused on developing the relations' meaning and language symmetry can eliminate the consistency effect. Our results show that explicit instruction seems to be necessary at all levels of education, especially in the period from 4th to 6th grade, regardless of the results of the research that imply that the development of understanding continues (spontaneously) in adolescence (Wassenberg et al. 2008). Second, a body of research focuses on the benefits of graphical representations of problem structures by using diagrams (Willis, Fuson 1988; De Koning et al. 2022) and expressing relations in different ways (Stern 1993; Boonen, Jolles 2015; Schumacher Fuchs 2012; Riley et al. 1983; Riley, Greeno 1988; Resnick 1983; Okamoto 1996; Okamoto, Case 1996). These representations could be used to improve students' understanding and achievement and reduce the consistency effect at higher levels of education.

The analysis of students' incorrect responses also supports the conclusion that students did not develop a conceptual understanding of the relations needed for solving CL and IL compare problems (Hegarty, Mayer, Green 1992; Hegarty, Mayer, Monk 1995; Stern 1993; Verschaffel 1994; Verschaffel, De Corte, Pauwels 1992; Lewis, Mayer 1987; Lewis 1989; Pape 2003; Van der Schoot et al. 2009). Several students solved the CL problem by adding all the numbers in the text of the problem. Interestingly, this kind of incorrect response was seen more often in 2nd grade (11.4%, Table 6) and 6th grade (9.5%) than in 4th grade (4.2%).

This kind of reasoning is described in the literature as "compute first and think later" (Stigler, Lee, Stevenson 1990: 15), keyword method (Briars, Larkin 1984), or number grabbing (Littlefield, Rieser 1993). If the students read the term *more*, they would respond by adding two numbers, without considering the context of the situation. It is also interesting that students gave fewer incorrect responses of this kind on the IL problem (4.5%, 0%, 4,8%, respectively, in 2nd, 4th and 6th grade, Table 6).

As was expected, the greatest number of incorrect responses to the IL problem was rooted in the relational term. Half of the 2nd graders (50%, Table 6) made an error in the relational term, while slightly less than a third of 4th and 6th graders made this mistake (31% and 26.2%, respectively, Table 6). One of the examples is shown in Picture 1.

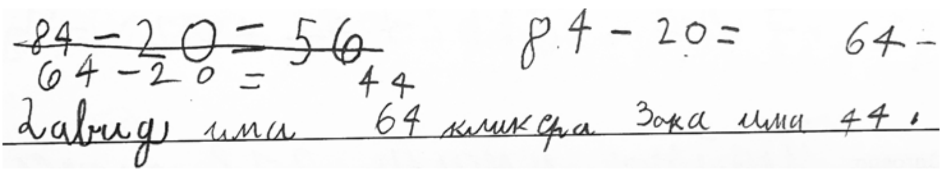
Picture 1. The incorrect relational term in students' responses



The combine-compare problem with a more complex structure (Problem C) is, in our opinion, cognitively challenging for 2nd graders. However, students in the 4th grade, especially in the 6th grade, should have a well-organized and flexible knowledge base that implies conceptual (e.g., using schematic representations for different types of problems) and procedural knowledge (formal and informal problem-solving strategies). Our results indeed show the statistical difference in the achievement of the 2nd, 4th, and 6th graders on this problem (Table 8), but, surprisingly, the success rate is low – slightly less than 10% of 2nd graders, slightly less than 30% of 4th graders, and slightly less than 60% of 6th graders solved the task correctly (Table 7).

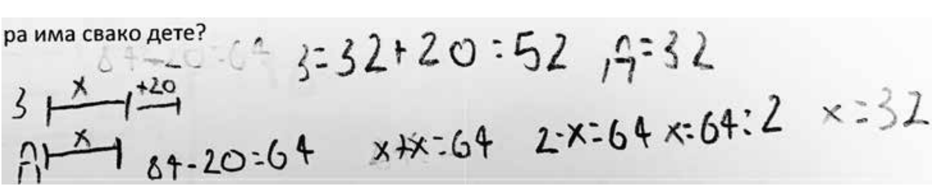
As expected, the incorrect responses show that the second graders had more difficulties with the problem with complex structure (77.3% of incorrect responses, Table 7) than students in 4th and 6th grade. They mostly tried to solve this problem using the keyword method (Briars, Larkin 1984) (38.6% of students, Table 9), as presented in the Picture. The 4th and the 6th graders mostly realized that the keyword method would not bring them to the correct solution; only 8.3% and 7.1% of students tried this method (Table 9).

Picture 2. Keyword method in solving problem with more complex structure

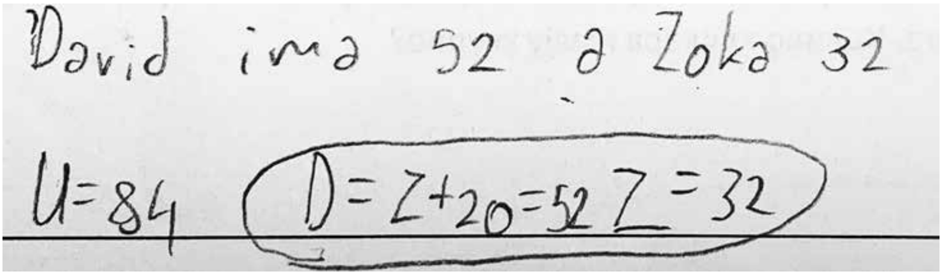


The analysis of students' strategies showed that only one student used graphical representation to solve the problem. This was not surprising for us because our previous research showed similar results (Zeljić, Dabić Boričić, Maričić 2021). On the other hand, it is surprising that only a few students used algebra to solve the problem – two of them in 4th grade and one in 6th grade (Picture 3). Two more 6th graders used algebraic symbols to represent relations in the problem, but they continued to solve it with arithmetic (Picture 4). Khng and Lee (2009) already noticed that many students return to arithmetic strategies of solving even if it was explicitly stated to solve the problem using equations. They consider using algebra for problem solving as moving forward to higher mathematics. Hence, students need to practice algebra even if they know how to solve the problem with the arithmetic method. We expected that 6th graders familiar with algebraic syntax and equation solving methods would use algebraic strategies for solving the problem with a more complex structure. In this context, the persistence in using arithmetic strategies could be considered an inhibition for further algebra learning.

Picture 3. Algebraic strategy and graphical representation in solving problem with more complex structure

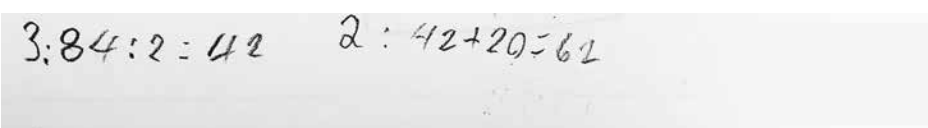


Picture 4. Recognized relations without algebraic strategy



For solving the problem with a more complex structure, students used two arithmetic strategies: 1) the one in which solving starts from the equal sets (computing 84: 2) and moves to the difference between them (by adding and subtracting 10); and 2) the strategy that starts from subtracting the difference and then making two equal sets. The second strategy was the strategy that led to the correct solution in greater numbers than the first one (Table 11). On the other side, some students who used the first strategy made one characteristic type of incorrect response. They started by making equal sets (dividing the total number of elements by 2), then added 20 to one set (Picture 5). They did not notice that the total number of elements does not fit the situation described in the problem. This solution shows that students do not have a coherent mental representation of all relevant elements and relations from the text of the problem (Hegarty, Mayer, Monk 1995; Pape 2003; Van der Schoot et al. 2009; De Koning et al. 2017; Koedinger, Nathan 2004) and that they do not apply modeling processes (Schwarzkopf 2007; Blum, Leiss 2007).

Picture 5. Incorrect the 'start from equal sets' strategy in solving problem with more complex structure



We can say that we confirmed the first hypothesis: students are more successful in solving problems with consistent language formulation. The second hypothesis is disproved: 1) There was no significant relationship between students' achievement on the CL problem and students' level of education; 2) We did not find differences in achievement of 4th and 6th graders on the IL problem (4th graders solved IL and CL better than 6th graders). The third hypothesis was related to solving the problem with a more complex structure, and it is confirmed: there was a significant difference in the achievement between 2nd, 4th, and 6th-grade students. Contrary to expectations, 6th graders did not use algebraic strategies in solving the problem with a more complex structure (4th hypothesis), and there are no differences in students' choice of strategy depending on their level of education. The most frequent mistake was the mistake in the understanding of relational terms, and it was based on keyword strategy.

CONCLUSION

In this research, we investigated achievement on and strategies for solving problems in which relational terms and language consistency are important. We looked into the possible effect of age/grade on achievement on three problems: the CL problem, the IL problem, and the problem with complex structure, which integrates simpler compare and combine word problems. Our results are in accordance with previous research, namely that students' achievement is better on the CL problem (with no difference between grades) than on the IL problem. There is a statistically significant difference in achievement between 2nd grade and 4th and 6th grade. However, there is no difference between 4th and 6th grade, which implies that there is a need for instructional intervention regarding understanding relational terms and problem-solving strategies. The most common strategy that led to an incorrect solution was the superficial strategy of using the keyword method. Students showed low achievement on the problem with more complex structure, though there is a significant difference between grades, with 6th graders being the best. Two solving strategies for this task stand out, one being the 'start from equal sets' strategy and the other 'start from the difference between sets' strategy, out of which the second strategy led to correct solution in more cases. Surprisingly, even though this problem is suitable for algebraic solving strategy or using diagrams, very few students used algebraic strategy, and only one student used graphical representation to solve the problem.

Understanding the problem-solving process is a very complex issue. Awareness of the different aspects of understanding and solving text problems can help us identify students' obstacles when trying to solve them. Instructions for understanding compare problems, which are based on verbal instructions and the use of diagrams and schemes, are still being developed and have not been implemented in educational practice. Several researchers have argued that the stereotypical nature of word problems in traditional textbooks encourages students to use superficial solving strategies, such as the keyword approach, without building an adequate model of the situation described in the problem. Students need rich experience with different semantic structures of tasks. The nature and structure of problems affect how students reason and can limit or expand understanding of mathematical concepts. Only the systematic use of all types of tasks and the planning of the solving process as an application of mathematical modeling leads to the ability of students to solve different types of mathematical tasks.

REFERENCES

Blum, Leiss (2007). W. Blum, D. Leiss, How do students and teachers deal with modelling problems, In: C. Haines, P. Galbraith, W. Blum, S. Khan (Eds.), *Mathematical modeling: Education, engineering, and economics*, Chichester: Horwood, 222–231.

Boesen, Helenius, Lithner, Bergqvist, Bergqvist, Palm, Palmberg (2014): J. Boesen, O. Helenius, J. Lithner, E. Bergqvist, T. Bergqvist, T. Palm, B. Palmberg, Developing mathematical competence: From the intended to the enacted curriculum, *Journal of Mathematical Behavior*, 33, 72–87.

Boonen, Jolles (2015): A. J. H. Boonen, J. Jolles, Second Grade Elementary School Students' Differing Performance on Combine, Change and Compare Word Problems, *International Journal School and Cognitive Psychology*, 2(122), 1–6.

Briars, Larkin (1984): D. J. Briars, J. H. Larkin, An integrated model of skill in solving elementary word problems, *Cognition and Instruction*, 1, 245–296.

Carpenter (1986): T. P. Carpenter, Conceptual knowledge as a foundation for procedural knowledge: Implications from research on the initial learning of arithmetic, In: J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*, Hillsdale, NJ: Erlbaum, 113–13.

Carpenter, Hiebert, Moser (1981): T. P. Carpenter, J. Hiebert, J. M. Moser, Problem structure and first-grade children's solution processes for simple addition and subtraction problems, *Journal for Research in Mathematics Education*, 12, 27–39.

Carpenter, Moser (1984): T. P. Carpenter, J. M. Moser, The acquisition of addition and subtraction concepts in grade one through three, *Journal of Research in Mathematics Education*, 15, 179–202.

Cummins (1991): D. Cummins, Children's interpretation of arithmetic word problems, *Cognition and Instruction*, 8, 261–289.

Cummins, Kintsch, Reusser, Weimer (1988): D. Cummins, W. Kintsch, K. Reusser, R. Weimer, The role of understanding in solving word problems, *Cognitive Psychology*, 20, 405–438.

De Corte, Verschaffel (1986): E. D. De Corte, L. Verschaffel, *Eye-Movement Data as Access to Solution Processes of Elementary Addition and Subtraction Problems*. Retrieved in January 2022 from https://files.eric.ed.gov/fulltext/ED273450.pdf.

De Corte, Verschaffel, De Win (1985): E. De Corte, L. Verschaffel, L. De Win, Influence of rewording verbal problems on children's problem representations and solutions, *Journal of Educational Psychology*, 77, 460–470.

De Corte, Verschaffel, Pauwels (1990): E. De Corte, L. Verschaffel, A. Pauwels, Influence of the semantic structure of word problems on second graders' eye movements, *Journal of Educational Psychology*, 82, 359–365.

De Koning, Boonen, Jongerling, Van Wesel, Van der Schoot (2022): B. B. De Koning, A. J. H. Boonen, J. Jongerling, F. Van Wesel, M. Van der Schoot, Model method drawing acts as a double-edged sword for solving inconsistent word problems, *Educational Studies in Mathematics*, 111, 29–45.

De Koning, Boonen, Van der Schoot (2017): B. B. De Koning, A. J. H. Boonen, M.Van der Schoot, The consistency effect in word problem solving is effectively reduced through verbal instruction, *Contemporary Educational Psychology*, 49, 121–129.

Fuson (1992): K. C. Fuson, Research on whole number addition and subtraction, In: D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, New York: Macmillian, 243–275.

Hegarty, Mayer, Green (1992): M. Hegarty, R. E. Mayer, C. E. Green, Comprehension of arithmetic word problems: Evidence from students' eye fixations, *Journal of Educational Psychology*, 84, 76–84.

Hegarty, Mayer, Monk (1995): M. Hegarty, R. E. Mayer, C. A. Monk, Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers, *Journal of Educational Psychology*, 87(1), 18–32.

Khng, Lee, (2009). K. H. Khng, K. Lee, Inhibition interference from prior knowledge: Arithmetic intrusion in algebra word problem solving, *Learning and Individual Differences*, 19, 262–268.

Kintsch (1988): W. Kintsch, The role of knowledge in discourse comprehension: A construction–integration model, *Psychological Review*, 95, 163–182.

Kintsch, Greeno (1985): W. Kintsch, J. G. Greeno, Understanding and solving word arithmetic problems, *Psychological Review*, 92, 109–129.

Koedinger, Nathan (2004): K. R. Koedinger, M. J. Nathan, The real story behind story problems: Effects of representations on quantitative reasoning, *Journal of the Learning Sciences*, 13(2), 129–164.

Lewis (1989): A. B. Lewis, Training students to represent arithmetic word problems, *Journal of Educational Psychology*, 81, 521–531.

Lewis, Mayer (1987): A. Lewis, R. Mayer, Students' miscomprehension of relational statements in arithmetic word problems, *Journal of Educational Psychology*, 74, 199–216.

Lithner, (2008): J. Lithner, A research framework for creative and imitative reasoning, *Educational Studies in Mathematics*, 67(3), 255–276.

Littlefield, Rieser (1993): J. Littlefield, J. J. Rieser, Semantic features of similarity and children's strategies for identification of relevant information in mathematical story problems, *Cognition and Instruction*, 11, 133–188.

Marzocchi, Lucangeli, De Meo, Fini, Cornoldi (2002): G. M. Marzocchi, D. Lucangeli, T. De Meo, F. Fini, C. Cornoldi, The Disturbing Effect of Irrelevant Information on Arithmetic Problem Solving in Inattentive Children, *Developmental neuropsychology*, 21(1), 73–92.

Morales, Shute, Pellegrino (1985): R. V. Morales, V. J. Shute, J. M. Pellegrino, Developmental differences in undersanding and solving simple word problems, *Cognition and Instruction*, 2, 41–57.

Mwangi, Sweller (1998): W. Mwangi, J. Sweller, Learning to solve compare word problems: The effect of example format and generating self-explanations, *Cognition and Instruction*, 16, 173–199.

Nesher, Greeno, Riley (1982): P. Nesher, J. G. Greeno, M. S. Riley, The Development of Semantic Categories for Addition and Subtraction, *Educational Studies in Mathematics*, 13(4), 373–394.

Nesher, Hershkovitz, Novotna (2003): P. Nesher, S. Hershkovitz, J. Novotna, Situation Model, Text Base and What Else? Factors Affecting Problem Solving, *Educational Studies in Mathematics*, 52(2), 151–176.

Okamoto (1996): Y. Okamoto, Modelling children's understanding of quantitative relations in texts: A developmental perspectives, *Cognition and Instruction*, 14, 409–440.

Okamoto, Case (1996): Y. Okamoto, R. Case, Exploring the microstructure of children's central conceptual structures in the domain of number, *Monographs of the Society for Research in Child Development*, 246 (61), 27–58.

Pape (2003): S. J. Pape, Compare word problems: Consistency hypothesis revisited, *Contemporary Educational Psychology*, 28, 396–421.

Powell, Fuchs, Fuchs, Cirino, Fletcher (2009): S. R. Powell, L. S. Fuchs, D. Fuchs, P. T. Cirino, J. M. Fletcher, Do word problem features differentially affect problem difficulty as a function of students' mathematics difficulty with and without reading difficulty?, *Journal of Learning Disabilities*, 42, 99–110.

Resnick (1983): L. B. Resnick, A developmental theory of number understanding, In: H. P. Ginsburg (Ed.), *The development of mathematical thinking*, New York: Academic, 109–151.

Riley, Greeno (1988): M. S. Riley, J. G. Greeno, Developmental analysis of language about quantities and of solving problems, *Cognition and Instruction*, 5, 49–101.

Riley, Greeno, Heller (1983): M. S. Riley, J. G. Greeno, J. H. Heller, Development of childrens problem solving ability in arithmetic, In: H. P. Ginsburg (Ed.), *The development of mathematical thinking*, New York: Academic Press, 153–196.

Schroeder, Lester (1989): T. L. Schroeder, F. K. J. Lester, Developing understanding in mathematics via problem solving, In: P. R. Trafton (Ed.), *New directions for elementary school mathematics*, Reston, VA: NCTM, 31–42.

Schumacher, Fuchs (2012): R. F. Schumacher, L. S. Fuchs, Does understanding relational terminology mediate effects of intervention on compare word problem?, *Journal of Experimental Child Psychology*, 111, 607–628.

Schwarzkopf (2007): R. Schwarzkopf, Elementary modelling in mathematics lessons: The interplay between "real-world" knowledge and "mathematical structures", In: W. Blum, P. L. Galbraith, H. W. Henn, M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI Study*, New York: Springer, 209–216.

Stern (1993): E. Stern, What makes certain arithmetic word problems involving the comparison of sets so difficult for children, *Journal of Educational Psychology*, 85, 7–23.

Stigler, Lee, Stevenson (1990): J. W. Stigler, S.-Y. Lee, H. W. Stevenson, *Mathematical knowledge of Japanese, Chinese, and American elementary school children*, Reston, VA: National Council of Teachers of Mathematics.

Van der Schoot, Bakker Arkema, Horsley, Van Lieshout (2009): M. Van der Schoot, A. H. Bakker Arkema, T. M. Horsley, E. D. C. M. Van Lieshout, The consistency effect depends on markedness in less successful but not successful problem solvers: An eye movement study in primary school children, *Contemporary Educational Psychology*, 34(1), 58–66.

Van Dooren, De Bock, Vleugels, Verschaffel (2010): W. Van Dooren, D. De Bock, K. Vleugels, L. Verschaffel, Just answering … or thinking? Contrasting pupils' solutions and classifications of missing-value word problems, *Mathematical Thinking and Learning*, 12(1), 20–35.

Verschaffel (1994): L. Verschaffel, Using retelling data to study elementary school children's representations and solutions of compare problems, *Journal of Research in Mathematics Education*, 25, 141–165.

Verschaffel, Van Dooren, Greer, Mukhopadhyay (2010): L. Verschaffel, W. Van Dooren, B. Greer, S. Mukhopadhyay, Reconceptualising Word Problems as Exercises in Mathematical Modelling, *Journal für Mathematik–Didaktik*, 31(1), 9–29.

Verschaffel, De Corte, Pauwels (1992): L. Verschaffel, De Corte, A. Pauwels, Solving compare problems: An eye movement test of Lewis and Mayer's consistency hypothesis, *Journal of Educational Psychology*, 84, 85–94.

Verschaffel, Depaepe, Van Dooren (2014): L. Verschaffel, F. Depaepe, W. Van Dooren, Word problems in mathematics education, In: S. Lerman (Ed.), *Encyclopedia of mathematics education*, Dordrecht, the Netherlands: Springer, 641–645.

Wassenberg, Hurks, Hendriksen, Feron, Meijs, Vles, Jolles (2008): R. Wassenberg, P. P. Hurks, J. G. Hendriksen, F. J. Feron, C. J. Meijs, J. S. Vles, J. Jolles, Age-related improvement in complex language comprehension: results of a cross-sectional study with 361 children aged 5 to 15, *Journal of Clinical and Experimental Neuropsychology*, 30(4), 435–48.

Willis, Fuson (1988): G. B. Willis, K. C. Fuson, Teaching children to use schematic drawings to solve addition and subtraction word problems, *Journal of Educational Psychology*, 80, 192–201.

Zeljić, Dabić Boričić, Maričić, (2021): M. Zeljić, M. Dabić Boričić, S. Maričić, Problem Solving in Realistic, Arithmetic/algebraic and Geometric Context, *Education and science*, 46(208), 413–430.

Маријана Ж. Зељић

Милана М. Дабић Боричић

Светлана М. Илић

Универзитет у Београду
Учитељски факултет
Катедра за методику математике

РАЗУМЕВАЊЕ РЕЛАЦИОНИХ ТЕРМИНА У ПРОБЛЕМИМА ПОРЕЂЕЊА-КОМБИНАВАЊА НА РАЗЛИЧИТИМ НИВОИМА ОБРАЗОВАЊА

Резиме: Једно од централних истраживачких питања у математичком образовању последњих деценија било је питање употребе текстуалних проблема са реалистичним контекстом, пошто у настави математике имају широку примену. У овом раду бавимо се *рутинским проблемима* који могу доприносити развоју концептуалног знања о основним рачунским операцијама. Определили смо се за испитивање успешности ученика на проблемима који настају интеграцијом проблема комбиновања и поређења. Иако су обе врсте проблема темељно истражене у литератури, нисмо наишли на истраживања која се баве успешношћу ученика на интегрисаним

проблемима, иако су овакви проблеми присутни у уџбеницима и наставној пракси. Њихов значај је, осим што доприносе концептуалном разумевању операција, у томе што могу указати на ниво развијености релационе терминологије код ученика, стратегије решавања проблема, као и на спремност за употребу графичких репрезентација. Интегрисани проблеми комбиновања и поређења могу бити различитих језичких формулација и нивоа комплексности. Проблеми поређења у овој интеграцији могу бити конзистентне и неконзистентне језичке формулације, док проблеми комбиновања могу допринети усложњавању структуре проблема. Стога смо истраживали успешност ученика на три различита типа проблема: 1) проблем са конзистентном језичком формулацијом, 2) проблем са неконзистентном језичком формулацијом, 3) проблем са сложенијом структуром. Претходна истраживања, која су се бавила истраживањем појединачних типова проблема, показала су да ученици имају највише потешкоћа у решавању проблема поређења, као и да су успешнији у решавању проблема поређења са конзистентном у односу на неконзистентну језичку формулацију. Резултати представљени у овом раду су део већег истраживања чији је циљ да испита да ли се релациона терминологија (термини „за толико више" и „за толико мање") развија са нивоом математичког образовања ученика и да утврди да ли овај развој прати и већи успех у решавању проблема са комплекснијом структуром. Стога наш узорак чине ученици другог, четвртог и шестог разреда. Истраживачка питања на која одговарамо у овом раду односе се на разлике у постигнућима ученика (другог, четвртог и шестог разреда) на интегрисаним проблемима са конзистентном и са неконзистентном језичком формулацијом, на везу између успешности ученика у решавању ових проблема и њиховог узраста (нивоа математичког образовања), на везу између успешности ученика у решавању задатка са комплекснијом структуром и њиховог узраста, као и на стратегије и честе грешке при решавању ових проблема. Узорак у истраживању су чинили ученици школа у Београду, и то 44 ученика другог, 48 ученика четвртог и 42 ученика шестог разреда. Резултати су потврдили резултате претходних истраживања – да су ученици успешнији у решавању проблема са конзистентном него са неконзистентном језичком формулацијом. Интересантан је резултат да нема разлике у успешности у решавању задатка са конзистентном језичком формулацијом између ученика другог, четвртог и шестог разреда – на целом узорку просечна успешност у решавању овог задатка је око 85%. То значи да око 15% ученика на свим истраживаним нивоима образовања имају потешкоће са разумевањем релационе терминологије у њеној најједноставнијој језичкој формулацији. Резултати су такође показали да на проблему са неконзистентном језичком формулацијом не постоје разлике у успешности између ученика четвртог и шестог разреда, што може упућивати на застој у развоју разумевања релационе терминологије након увођења алгебре у математичко образовање, а самим тим и на потребу за више инструкционих интервенција у на овом узрасту. Резултати ученика на задатку са сложенијом структуром показали су да постоје разлике у успешности ученика на различитим нивоима образовања. Очекивано, најмање успешни су били ученици другог разреда, затим четвртог, док су најуспешнији били ученици шестог разреда. Очекивано, ученици другог и четвртог разреда нису користили алгебарске стратегије решавања, а изненађујуће је да ни ученици шестог разреда нису користили алгебарске стратегије. Овај резултат потврђује мишљење многих аутора да треба инсистирати и на алгебарским стратегијама решавања проблема иако ученици умеју да га реше аритметичком стратегијом. Анализом одговора ученика на овај задатак препознато је да је „метод кључне речи" најчешће водио ученике ка нетачном решењу. Такође су препознате и две стратегије решавања проблема – она која „полази од једнаких скупова" и она која „полази од разлике међу скуповима", при чему је друга стратегија у већем броју случајева водила према тачном решењу. Такође, ученици нису корстили сликовне репрезентације у решавању овог проблема, иако је проблем био погодан за њихово коришћење. Истраживачи су раније приметили да стереотипско коришћење текстуалних проблема у традиционалним уџбеницима подстиче ученике да користе површинске стратегије решавања, као што је метод кључне речи. Стога је потребно обогатити искуства ученика са проблемима различите семантичке структуре, чиме се утиче на процес њиховог мишљења и разумевања математичких концепата. Систематском употребом свих врста задатака и стратегија које примењују процес математичког моделовања може се утицати на побољшање постигнућа ученика у решавању свих типова математичких проблема.

Кључне речи: текстуални проблеми, проблеми комбиновања, проблеми поређења, стратегије решавања проблема, математичко образовање.

Jasmina V. Milinković
University of Belgrade
Teacher Education Faculty
Olivera S. Simić
Primary school “Borivoje Ž. Milojević”
Krupanj

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CHALLENGES OF SOLVING VISUALLY PRESENTED PROBLEMS

Abstract: Organization of the teaching process which enables the acquisition of quality and effective mathematical knowledge applicable in different life situations and which lays the foundation for lifelong learning is based on problem-solving. Teaching how to solve visually presented problems is one objective that contributes to these overall goals. The main research goal of this paper is an experimental examination of the effects of problem-based teaching in the development of mathematical modeling skills involving visually presented problems. The effectiveness of teaching visually presented problems for the development of mathematical modeling skills in solving equations and inequalities in the fourth grade of primary school is examined. The descriptive method was used for the analysis, processing, and interpretation of the research results to investigate the type of errors pupils make when dealing with the visually presented information. Participants in the experimental program showed a higher level of knowledge when solving simple and complex equations and inequalities as well as in composing texts based on given iconic representations of equations and inequalities, as opposed to the pupils not influenced by the experimental model. Given its positive effects on the development of mathematical modeling skills, teaching visually presented problems is justified during the early years of mathematics education.

Keywords: visualization, problem posing, problem-solving, mathematical modeling, equation and inequality.

“We live in a world where information is transmitted mostly in visual wrappings” (Arcavi 2003: 215). Mathematics textbooks for grades K to 4 of elementary school are full of illustrations. Iconic representations are used to present mathematical concepts such as numbers, arithmetic operations, arithmetic laws, etc., often using multiple iconic representations to represent the same concept. Yet, symbolic or language representations are dominant in math problem formulations. School practice shows that mathematical terminology and symbolism tend to represent an obstacle to understanding abstract mathematics concepts or solving math problems (Roubiček 2007). Semiotic analysis can shed light on problems since the problems tend to be caused by different mental representations of the teacher and the pupil’s non-conventional writing forms (Ibid.). Visual representations are seen

jasmina.milinkovic@uf.bg.ac.rs

as a mediator between pupils' meta-language and mathematics language. Iconic representations, such as drawings, tables, and graphs are used in problem formulations. The question of interest to us is if pupils know how to interpret a visually presented problem.

VISUALIZATION IN PROBLEM POSING

The idea that visual representation is a tool in math reasoning tools is well documented in the literature (Janvier Dufour 1987; Kaput 1987; Cobb et al. 1992; Lesh 1981; Cuoaco, Curcio 2001; Michalewicz, Fogel 2000; Reed Woleck 2001). "Real-world meanings can be acted out, modeled with objects, and drawn with simplified math drawings" (Fuson 2004: 118). "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, to depict and communicate information, thinking about and developing previously unknown ideas and advancing understandings" (Arcavi 2003: 217). According to Norma Presmeg (2006), visualization is taken to include processes of constructing and transforming both visual and mental imagery and all of the inscriptions of a spatial nature that may be involved when doing mathematics. She concentrates on visual constructs as aids in the formation of mental concepts. Arcavi, on the other hand, attends to the role of visualization in mathematical reasoning in problem-solving and problem posing. We consider visualization as a synonym for an iconic representation.

Arcavi believes that seeing things sharpens our understanding and triggers questions that we would not pose otherwise. If we recognize that visualization offers a link between the real world and abstract mathematical concepts it becomes obvious that we should create opportunities for pupils to investigate visually presented problems. If we recognize that visualization offers a link between the real world and abstract mathematical concepts it becomes obvious that we should create opportunities for pupils to investigate visually presented problems. Nicole Venuto and Lynn C. Hart (2017) demonstrated in an investigation the change in the children's ability to explain their mathematical reasoning in writing combined with drawing, suggesting that adding writing to conceptually based lessons improves children's ability to communicate their thinking using visual tools. One way to help pupils become confident in using visualization to problem-solve is to confront them with problems posed in visual forms. Researchers emphasize the importance of problem visualization as a link between external representations and mental images (Singer et al. 2011; Wittmann 2005; Milinkovic, in print). Friedlander and Tabach (2001) argue that using multiple representations of problems provides models which can be used in the problem-solving process. Singer and colleagues (2011) pointed out that the task format underlines the sequence of transfers from external to an internal representation. Barwise and Etchemendy claim that mathematicians value diagrams and other visual tools both for teaching and in the process of mathematical discovery. But despite the obvious importance of visual images in cognition, visual representation remains undervalued in the theory and practice of mathematics. Proofs based on diagrams or graphs are considered to be informal and "half-proved". However, there is a strong push toward the other direction. Nowadays, visual forms of representation are considered to be "legitimate elements of mathematical proofs." (Barwise, Etchemendy, In: Arcavi 2003: 226).

Research shows that pupils have difficulties in dealing with visually possessed problems. Lucia Csachová and Mária Jurečková (2019) found that 10 and 11-year-old pupils undergoing the Slovak nationwide testing have had difficulties linked to their inability to read data from figures. On the other hand, some selected problems were easy for pupils because figures (e.g. illustrations) made the problem-solving process easier (Ibid.). Christian Rütten and Stephanie Weskamp (2019) designed a combinatorial learning environment for fostering reasoning skills: diagrammatic reasoning, conjectures, and justifications in building towers of cubes and cuboids. They stated that diagrammatic reasoning synthesizes "the construction/observation of a diagram, the observation of structural relations among its parts, and the perceptual manipulation and thought-experimentation to infer new possible relations conducive to the attainment of the conceptualization of the Object of the sign-vehicle (the Object-as-it-is)" (Rütten, Weskamp 2019: 364). They point out that the aim of diagrammatic reasoning is the construction of a mathematical argument that warrants the abstract structure of the mathematical object, whereas visualization mediates the emergence of diagrammatic reasoning to arrive at generalization (Ibid.).

Knowledge of representations as was noted earlier is particularly important in problem-solving (Polya 1957; Goldin 1987). Friedlander and Tabach (2001) argue that the teacher's presentation of problem situations with different representations encourages flexibility in pupils' choice of representations. They state that "the presentation of a problem in several representations gives legitimatization to their use in the solution process" (Ibid.: 176). But in practice, teachers rarely consider different representations of problem posing as an important issue.

Sofia Anastasiadou (2009) presents a structural equation model representing the hierarchical structure of translation among representations in frequency concept to 6th-grade Greek pupils. The researcher aimed to contribute to the understanding of the approaches that pupils use in solving tasks related to the frequency concept and to examine which approach is more strongly correlated to their success in such tasks. The structural model that resulted from the analysis of the data confirms the existence of five first-order factors relative to frequency representations. Barbora Divišová and Naďa Stehlíková (2011) report on research that deals with a certain type of geometric problem, i.e., problems effectively solvable without algebraic calculations, for which they observed pupils' preference for (often lengthy) algebraic solutions over (often quick) geometric ones.

VISUALIZATION PROBLEMS IN MATHEMATICS MODELING

Mathematical models, which are results of mathematical modeling, emphasize the structural properties and functional relationships of real-life objects or situations (Lehrer, Schauble 2003, 2007; Lesh, Doerr 2003). "Mathematical model is a formal mathematical record that reflects aspects of the studied phenomenon, often in the form of a graph, equation or algorithm" (Milinković 2014: 46). Also, the mathematical model can be in the form of a diagram (Galbraith, Stillman, according to Barbosa 2006). The model serves as a means of mediating between the real world and the abstract world of mathematics. In other words, models help pupils to solve a problem at every level of abstraction (Milinković 2016). It is used to construct, describe, and interpret certain mathematical situations (Richardson 2004). Terwel and colleagues see in the model "a certain structural form of representation" where representation is a broader and more comprehensive term from cognitive psychology, and model is a term used in mathematics education (Terwel et al. 2009: 27). However, some authors understand the model as a representation of the task that is created and formed, intending to summarize and understand the essence of the tasks (Novick, Bassok 2005).

For mathematical modeling to manifest its positive role in the teaching process, it is necessary to engage teachers in terms of encouraging and developing the following student skills: (1) interpretation of mathematical or scientific phenomena and information presented in the form of text or diagrams; (2) understanding, analyzing, and reading simple examples of tabular data; (3) collection, analysis, and interpretation of data; (4) preparation and compilation of written reports based on the analyzed data; (5) communicating in a group and working together on data; (6) construct models with the group through verbal and written reports (Watters et al. 2000). As it was stated, understanding, interpreting, analyzing, and using diagrams or tabular data are essential in the process of mathematical modeling (Ibid.). The ability to manipulate various representations and use them in a novel problem is considered to be of critical importance in problem-solving (Obradovic, Zeljic 2015).

METHODOLOGY

This paper reports partial results of a larger study investigating the effects of implementing problem-based instruction on pupils. One *objective* of the study was to examine how successful pupils are in composing text tasks related to equations and inequalities based on assigned iconic and symbolic representations, after conducting problem-based teaching. We have determined that the dependent variable was pupils' test scores. The independent variable is the level of mastery of the methods of solving equations and inequalities under the influence of problem-based teaching (model to be introduced).

Research methods. The research is a combination of experimental and descriptive scientific research methods, as well as a theoretical analysis method. We compared the effects of problem-based teaching. Here we particularly explored the elements of research dealing with the development of pupils' ability to understand and use visually wrapped information in the process of problem-solving (Arcavi 2003). The study is an experiment with parallel groups. The experimental program was introduced to each of the two experimental groups, while teaching in the two control groups was carried out in a traditional (classical, established) way. The traditional way of teaching is based on the dominant, lecturing role of the teacher who delivers and transfers ready-made knowledge to the student (Woodlief 2007). Through the frontal form of work with the student, the teacher achieves one-way communication, which disrupts all forms of interaction. In the traditional approach, there is not enough time for the student to engage in independent activities (Bognar, Matijević 2002). *The sample* consisted of pupils of the experimental and control groups, namely 88 pupils who attended the 4th grade of an elementary school in Krupanj and Loznica. The sample has elements of random and cluster sampling. We randomly selected primary schools for the experimental and control groups. An equal number of participants were selected for both groups.

Description of research activities. The research was carried out through multiple steps over 2 months. Here we focus on the results obtained from the initial test and the final test in which we determined the effects of problem-based teaching on the development of mathematical modeling abilities when solving equations and inequalities. With the students of the experimental group (44 students), a total of six lessons related to solving equations and inequalities were realized. In order to check the effects of problem-based teaching in the development of mathematical modeling abilities, when solving equations and inequalities, a testing technique was applied. When conducting the experiment, there were conditions that enabled the presence of researchers in both groups (students of group E attended classes in the morning shift, while students of group K attended classes in the afternoon shift) with the intention of obtaining the most valid results. In both groups, an initial test was conducted first, followed by two final tests, where the first was about solving equations and inequalities, and the second was about understanding the idea of a function. Considering the goal of our paper, we reduced the exposition of the results of the quantitative analysis to a minimum, devoting space to the qualitative analysis of the pupils' answers.

RESULTS

Based on the research results (Tables 1–6) a quantitative analysis was carried out and served as a starting point for performing qualitative analysis.

Table 1. Descriptive characteristics of the sample for all variables

Variables	Number of pupils	AS	S	MIN	MAX
Initial test	88	19.27	4.85	8	30
Final test	88	21.52	10.37	3	38
Complex equations and inequalities	88	3.16	1.73	0	8
Simple equations and inequalities	88	14.47	3.29	6	22
Problem formulation based on iconic and symbolic representations	88	1.63	1.15	0	3

Legend: AS – Arithmetic mean; S – Standard deviation; MIN – minimum result: MAX – maximum result

During the analysis of the initial test, we became familiar with certain subcategories in the tasks with which the pupils of neither the E nor the K groups were able to cope. Looking at Table 2, positive values of asymmetry show that most of the obtained results are to the left of the mean, among smaller values, and negative values of asymmetry show that most of the results are to the right of the mean, among larger values.

Table 2. Elementary statistics for Groups E and K

	Variables	AS	S	MIN
Experimental group (E) N=44	Initial test	19.05	5.14	8
	Final test	30.09	5.74	14
	Problem formulation based on iconic and symbolic representations	1.68	1.19	0
Control group (K) N=44	Initial test	19.50	4.59	8
	Final test	12.95	5.87	3
	Problem formulation based on iconic and symbolic representations	1.57	1.11	0

In the final test, tasks f1a3 and f1b3 are parallel versions of task i2a3 that the pupils solved in the initial test and were related to solving simple equations where it was necessary to present the given equations in an iconic way. Table 3 indicates that the implementation of the introduced Model (Experimental Program) was of great importance for the success of pupils in the experimental group.

Table 3. Performance of E and K groups during the iconic representation of simple equations on the Final Test

Iconic representation of simple equations (h – 430 = 2350, 560 : h = 70)			
	N	f $1a_3$	f $1b_3$
E group	44	97.72%	56.82%
K Group	44	4.54%	0%

The E group was significantly more successful than the K group, which demonstrates the fact that the pupils, with the help of the introduced Model, managed to overcome the difficulties they encountered in the initial test in a task with the same requirements. We think that the low performance of pupils of the K group stems from the fact that the pupils simply did not encounter the iconic representations of equations, but also that they have not developed enough mathematical modeling ability. Also, one of the tasks that turned out to be difficult for pupils of both groups on the initial test was solving an inequality with the help of a table. We will see what the situation is after the introduced Model in Table 4.

Table 4. The success of pupils of E and K groups in solving the inequation using the table

Solving inequality using a table (25 · h < 200)		
	N	f $2a_3$
E group	44	97.72%
K Group	44	0%

Again, we are faced with the fact that the E group showed great interest in the Model that was introduced and thus demonstrated its high success in overcoming the problem it encountered in the initial test. On the other hand, the pupils of the K group simply ignored the table and solved the given inequality in the usual way using knowledge related to the expression of unknown components.

Table 5. Achievement of pupils of groups E and K in transposing a simple inequality from iconic to symbolic form

Transforming an iconic representation of inequality into a symbolic one		
	N	f $4a_1$
E group	44	97.72%
K Group	44	9.09%

The percentage difference between the control and experimental groups is really large, which again confirms the high efficiency of problem-based teaching in the development of mathematical modeling abilities. When solving tasks related to complex equations, we previously talked about the fact that the pupils of the

E and K groups had certain difficulties during their iconic presentation, then we introduced the Model to the experimental group to find out if the situation at that time would change. In the following text, we will see whether and to what extent the pupils developed the ability to do mathematical modeling and how successfully they were able to transpose equations from the iconic frame to the symbolic one.

Table 6. Success of pupils of groups E and K in transposing complex equations from iconic to symbolic form

Transforming an iconic representation of a complex equation into a symbolic one					
	N	f6 and$_1$	f9a$_1$	f10a$_1$	f10a$_3$
E group	44	95.45%	45.45%	77.27%	88.64%
K Group	44	45.45%	0%	6.82%	2.27%

Note that a few pupils from the E group simply completed the picture very easily, and based on the picture, they immediately came to the correct solution without setting the equation. They ignored the request for the symbolic notation of the equation; however, the final answer and solution were correct. The situation with the group (K), which was not influenced by the Model, was very different from the E group, because only 3 pupils correctly wrote the symbolic notation of the equation, and only one pupil completed the given picture. In the K group, an additional 2 pupils correctly presented the equation symbolically based on the realistic situation, ignoring the picture completely, and thus came to its solution.

To analyze the types of difficulties that pupils had when composing tasks based on visual or symbolic representations, we created subcategories: (a) representation of a symbolic record with an adequate iconic representation; (b) solving the inequality with the help of a table; (c) composing the appropriate text based on iconic and symbolic representations and (d) noticing the rules based on which a certain table was filled in, which was a reflection of the idea of the function. To confirm the above, with the help of the following Table 7, the success of solving tasks by pupils according to the specified subcategories can be seen.

Table 7. Percentage of pupils who successfully solved the tasks according to the specified subcategories on the initial test

		Transforming symbolic notation into an iconic representation		Solving the inequality using a table		Problem posing based on iconic and symbolic representations	Drawing conclusions (functional idea)
	N	i2a$_3$	i10a$_3$	and 5a$_3$	and 5b$_3$	i6a$_5$	i11a $_6$
E group	44	18.18%	9.09%	29.55%	22.73%	31.82%	2.27%
K Group	44	13.64%	6.82%	20.45%	15.91%	22.73%	2.27%

After the initial measurement, the experimental program was introduced to the experimental group, after which the first final measurement was carried out.

Table 8. Final test results

Competences in Mathematical modeling	Problem posing based on iconic and symbolic representations		Total (10 tasks)
Task	4.	6.	
E group	130	106	
	236		1366
K Group	11	37	
	48		569

Differences in the achieved success between the experimental and control groups at the final measurement are evident, where the experimental group shows a marked improvement both in relation to the initial measurement and in relation to the control group in the final measurement, while the control group shows a certain decline considering its initial state and a drastic drop compared to the group in which the experimental program was introduced (Table 8).

DISCUSSION

During the analysis and interpretation of the results of the initial test, one of the main difficulties faced by the pupils when solving the tasks was related to the composition of the relevant text with the given iconic and symbolic representations of the equations. In the final test, the pupils had this requirement in two tasks: task f4a5, where it was necessary to first compose the inequality we talked about earlier in symbolic form, based on the iconic representation of the inequality, and then come up with a text that describes it, and task f6a5, where it was necessary to compose the symbolic form of the equation based on the iconic representation and then compose the text task that corresponds to the given image (equation). Considering the results, we see that the E group achieved significantly better results compared to the control group in both cases. We believe that in this case as well, the Model that we applied to the E group stood out and proved to be very effective.

Mistakes made by pupils when solving tasks on knowledge tests, i.e. when emphasizing the degree of development of mathematical modeling skills, can be categorized according to the following groups:

- neglecting the structure of the expression (equation, inequality) in its iconic presentation;
- wrong transposition of the equation from the iconic to the symbolic environment;

• ignoring the iconic and symbolic form of expression (equations and inequalities) when designing the math task in textual form;

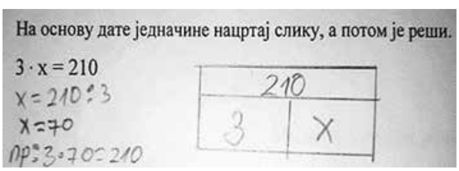
• misunderstanding the structure of the task textual formulation and mixing formulas;

• inadequate execution of rules and conclusions based on the visual display of the problem.

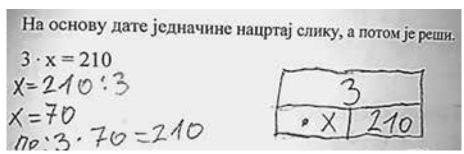
We will support each of the mentioned types of errors with adequate examples (pupils' work) that will complete our analysis. Given that the pupils solved the simple equation (3 · h = 210) very easily, the second task on the initial test gave them a lot of difficulties in the sense that it was necessary to convert the same equation from a symbolic form to an iconic. Many pupils neglected the structure of the set equation and were guided by their previous experience on the first task in which they also very easily supplemented the set iconic representations of the equations when it was necessary to sketch the picture corresponding to the equation themselves, errors occurred. How pupils represented an equation of the form 3 · x = 210 is illustrated in Example 1.

Example 1. Misunderstanding the structure of an expression (equation) when designing an iconic representation

1a) P.24E-i2 (P – pupil)



1b) P.81K-i2

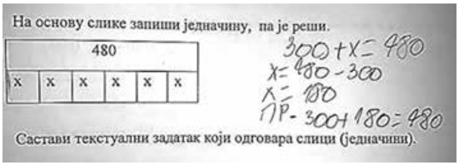


As the pictures show, the pupils successfully came to the solution of the given equation, but the iconic representation is not relevant to its symbolic record. We believe that the pupils did not pay attention to the structure of the equation and instead were guided by the equations related to addition and subtraction and their pictorial forms, thus coming to the wrong iconic representation of the simple equation 3 · x = 210. In both examples (Example 2a and b) it can be seen that the pupils need to include the given symbolic components of the equation in some way only in the table that they have previously sketched, without taking into account what the structure of the set of equations represents. This form of error appeared very often in the works of pupils of both E and K groups, and as such, we classified it in a special category of the most common mistakes made by pupils when solving tests.

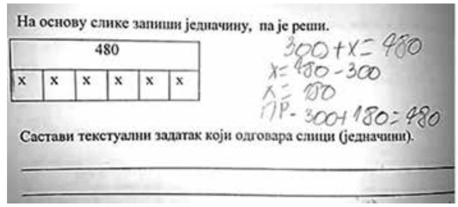
The next mistake refers to the situation in which the pupils underwent the task that examines the ability to transpose information from an iconic to a symbolic environment (pupils should represent the given equation symbolically) as an illustration that "beautifies" it. Symbolic representations of a given iconic representation are not relevant to it (Example 2).

Example 2. Misunderstanding the iconic form of the equation

2a) P.5E-i6



2b) P.70K-i6

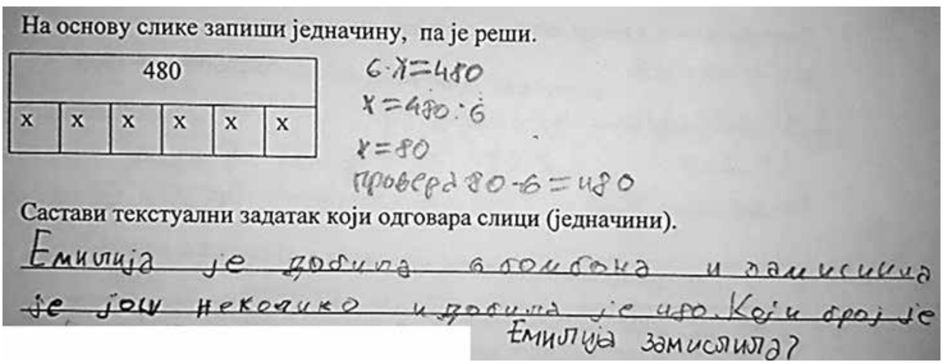


The activity of transposing information from an iconic to a symbolic environment caused a lot of difficulties for the pupils, and in connection with that, during the analysis of the papers, we came across three subcategories of this problem. Observing the work under a) P.35E-i6, it is seen that the student first does not understand the iconic structure of the equation of the form $6 \cdot x = 480$, and then writes down the symbolic structure of a mathematical expression that is meaningless ($x = 480 + x = 6$). In the second paper under b) P.5E-i6, the student shows that he somewhat understands the structure of the picture, but most likely due to carelessness when reading the instructions for solving the task, he writes down an inequality instead of an equation.

We see that this pupil unsuccessfully tried to compose the text (which will be discussed later), firstly because he did not write down an adequate equation based on the picture, and then due to carelessness when reading the second request, the student writes down a sentence that also sounds meaningless instead of a text task. The third paper, under c), also shows the pupil's misunderstanding of the iconic representation of the equation ($6 \cdot x = 480$) in which the student constructs the equation by adding elements that the equation does not contain in its iconic form. For unknown reasons, the student uses the number 300 as one of the components of an inadequately written equation and sees the set iconic representation as an equation of the form $y + x = z$, instead of $y \cdot x = z$. After all the above, we can conclude that the transposition of the equation from the iconic to the symbolic environment was very difficult for the pupils of the fourth grade of elementary school. However, the pupils may have encountered difficulties in this task because they did not have any or sufficient experience in modeling and symbolic representation of equations. The data we obtained after the introduction of the experimental program support the previously stated (pupils do not have adequate experience in modeling) because the pupils (E group) who were influenced by the Model showed that they overcame this type of problem and thus achieved a much better result than the control group (which was discussed earlier). We have grouped all the mentioned examples under one category, i.e., the wrong transposition of the equation from the iconic to the symbolic environment, because we believe that every type of problem mentioned is based on a misunderstanding of the iconic form of the equation. The next problem faced by the pupils is related to the situation in which the pupils ignore the iconic/symbolic form of the equation, and because of that, they were not able to come up with an adequate text task that would complete the given equation. In Example 4, we present work from a student to elucidate the previously mentioned difficulty.

Example 3. Ignoring the iconic/symbolic form of the equation when designing a word problem

3a) P.7E-i6

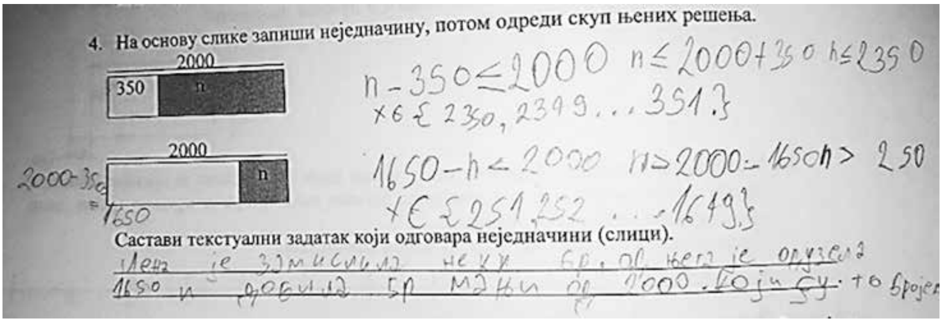


The student's solution shows that the student of the experimental group knows and understands the iconic representation of the equation, then correctly represents its symbolic notation, and arrives at the correct solution of the equation. However, the problem arose when it was necessary to compose an adequate text corresponding to the given equation. In this case, the student composes a task that is meaningless both in the textual and mathematical context. First, there was talk about "imaginary candies", while at the end of the tasks, those candies were reduced to "imaginary numbers". The student started the text with one idea and ended with another idea.

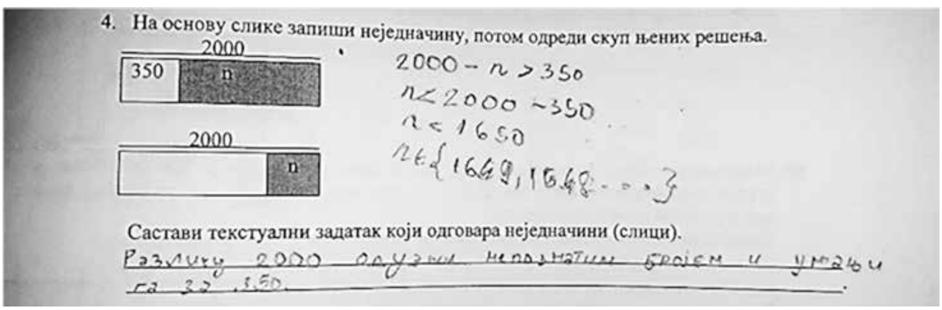
One gets the impression that the student paid attention to the symbolic representation of the equation because he mentioned the components of the equation in the text, but still during the process of writing the text he "lost his train of thought" and thus made the problem formulation meaningless. The problem that we classified in the same category as the previous one was related to composing a text task based on an iconic/symbolic representation, but this time with inequations.

Example 4. Ignoring the iconic/symbolic form of inequation when designing a word problem

4a) P.60K-f4



4b) P.27E-f4

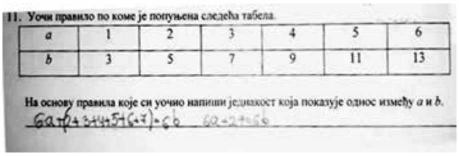


Example 4a (P.60K-f4) is the work of a student of the control group, based on which we can conclude that the student does not understand the iconic form of the given inequality, which creates the conditions for the further emergence of problems when composing the text task that was supposed to describe that inequality. Example 4b (P.27E-f4) shows that the student of the experimental group understands the iconic form of the inequality, writes it down correctly, solves it, and gives a set of solutions. In this example, the student shows that he understands the connection between the created pictorial representation of the inequality and its symbolic notation, which is not the case in Example 4a. However, the problem of composing the text was manifested again. It is possible that these results were obtained because the pupils' previous experiences related to such requests did not occur.

The last mistake, or rather the difficulty that the pupils encountered, is related to a thematic area that was outside the area of equations and inequalities. It was about the idea of function.

Example 5. Inadequate determination of the rules and dependencies between the sizes based on which the given table was filled

5a) P.62K-i11



5b) P.14E-i11

In both examples of this type of task, the pupils clearly show that they do not understand how to derive a rule based on the given data from the table, that is, the dependence between the variables a and b. They show that in their previous experience they have no traces of the idea of a function. This was the task on the initial test that was performed most unsuccessfully by the pupils of both groups.

CONCLUSIONS

In this paper, we addressed the role of mathematical visualization in problem posing. We have argued that mathematical visualization provides cognitive accessibility to problems. How useful visualization may be in mediating the pupils' mental passage from a realistic world to an abstract one as needed was investigated in the reported empirical study. Based on the evidence it was argued that visualization is an important pedagogical tool in mathematics teaching, as it provides a modality of reasoning. We also recognize that the evidence of research in model situations is limited the importance of the context of learning and teachers' expertise for student outcomes. We need to make an effort to strengthen pupils' ability to understand and deal with visually presented information. Teachers need to learn about the most common mistakes, doubts, and difficulties of pupils when dealing with visually presented information to have a realistic idea of student possibilities and achievements. We believe that this paper contributes to that. Difficulties translated into mistakes made by students when solving equations and equalities were classified as the following: (a) Misunderstanding of the structure of expressions (equations) when designing an iconic representation (Example 1); (b) Misunderstanding the iconic form of the equation (Example 2); (c) Ignoring the iconic/symbolic form of the equation when designing the text task (Example 3); (d) Ignoring the iconic/symbolic form of inequality when designing a text task (Example 4); (e) Inadequate determination of the rules and dependencies between the sizes based on which the given table was filled (Example 5). Namely, after the implementation of the experimental program, the results of this research showed that the students who learned content related to equations and inequalities through the implementation of problem-based teaching showed significantly fewer errors on the final knowledge test compared to the initial test, while the students of the control group made the same mistakes on the final test. Some of the requirements that modern educational practice puts before us can be realized through a series of recommendations that arise from our research: 1) the teaching process should be enriched with innovative models such as problem-based teaching; 2) continuously guide students to be as active participants in the teaching process as possible in terms of encouraging them to independently draw conclusions, approach problems from different points of view, model, exchange experiences and knowledge with other students, etc.; 3) students should be trained to connect different problem situations with the real world, so that they can more easily understand their abstraction; 4) pay more attention in the lessons in the field of equations and inequalities to the connection between iconic and symbolic representations in order to develop the ability of mathematical modeling. The importance of connecting problem situations with their real context was also demonstrated during the research conducted by Lazić and Milinković (2017). In their research, it was shown that students are very happy to approach solving problems related to fractions given in the form of pictures, tables, and graphs, while the symbolic presentation of fractions without an accompanying picture caused difficulties, which indicates a similarity with our results in this paper. Milinković, Mihajlović, and Dejić (2019) also conducted research that showed that students aged 11 can very successfully use different representations and models when solving mathematical problems. Namely, if students are properly trained to correctly connect iconic and symbolic representations, they will not have difficulties when transforming one into the other, which was the result of research in our E group after the introduction of the experimental program. By summarizing the entire analysis and interpretation of the results, we concluded that the implementation of problem-based teaching in mathematics classes effectively aids in the development of mathematical modeling abilities, particularly the development of competencies in dealing with visually presented data.

REFERENCES

Anastasiadou (2009): S. Anastasiadou, A Structural equation model representing the hierarchical structure of translations among representations in frequency concept, In: J. Novotná, H. Moraová (Eds.), *SEMT 2009*, *Development of mathematical understanding*, August 23–28, 2009, Prague, The Czech Republic: Charles University, Faculty of Education, 57–64.

Arcavi (2003): A. Arcavi, The role of visual representations in the learning of mathematics, *Educational studies in mathematics*, Netherlands: Academic Press, 52, 215–241.

Barbosa (2006): J. C. Barbosa, Mathematical modeling in classroom: a critical and discursive perspective, *ZDM*, 38(3), 293–301.

Bognar, Matijević (2002): L. Bognar, M. Matijevic, *Didaktika*, Zagreb: Školska knjiga.

Cobb, Yackel, Wood (1992): P. Cobb, E. Yackel, T. Wood, A Constructivist Alternative to the Representational View of Mind in Mathematics Education, *Journal for Research in Mathematics Education*, 23(1), 2–33.

Couco, Curcio (2001). A. Couco, F. R. Curcio (Eds.), *The roles of representation in school mathematics*. *NCTM Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

Csachová, Jurečková (2019): L. Csachová, M. Jurečková, Analysis of problems with figures in primary grades mathematics, In: J. Novotná, H. Moraová (Eds.), *Proceedings of SEMT 2019*, *Opportunities in Learning and Teaching Elementary Mathematics*, August 18–22, 2019, Prague, The Czech Republic: Charles University, Faculty of Education, 140–148.

Divišová, Stehlíková (2011): B. Divišová, N. Stehlíková, Geometric problems effectively solvable without algebraic calculations, In: J. Novotná, H. Moraová (Eds.), *Proceedings of SEMT 2011*, *The mathematical knowledge needed for teaching in elementary schools*, August 21–26, 2011, Prague, The Czech Republic: Charles University, Faculty of Education, 126–134.

Friedlander, Tabach (2001): A. Friedlander, M. Tabach, Promoting multiple representations of Algebra, In: A. Cuoco, F. Curcio (Eds.), *The roles of representations in school mathematics*, *2001 Yearbook*, NCTM: Macmillan Publishing Company, 173–186.

Fuson (2004): K. C. Fuson, Pre-K to grade 2 goals and standards: Achieving 21st-century mastery for all, In: D. H. Clements, J. Sarama (Eds.), *Engaging young children in mathematics: Standards for early childhood education*, Mahwah, NJ: Lawrence Erlbaum, 105–148.

Goldin (1987): G. Goldin, Cognitive representational systems for mathematical problem solving, In: C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, Hillsdale, NJ: Lawrence Erlbaum Associates, 125–145.

Janvier (1987): C. Janvier, *Problems of representation in the teaching and learning of mathematics*, Hillsdale, NJ: Erlbaum.

Kaput (1987): J. Kaput, Representation systems and mathematics, In: C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, Hillsdale, NJ: Erlbaum, 19–27.

Lazic, Milinkovic (2017): B. Lazic, J. Milinkovic, Using multiple representations of fractions to enhance problem solving, In: J. Novotna, H. Moraova (Eds.), *Equity and diversity in elementary mathematics education* (Proceeding of International Simposium Elementary Math Teaching), Prague, The Czech Republic: Charles University, Faculty of Education, 301–311.

Lehrer, Schauble (2003): R. Lehrer, L. Schauble, Origins and evloution of model-based reasoning in Mathematics and Science, In: H. Doerr, R. Lesh (Eds.), *Beyond constructivism: A models and modeling perspective*, Mahwah, NJ: Erlbaum, 59–70.

Lesh, Doerr (2003): R. Lesh, H. M. Doerr, Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving, In R. Lesh, H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*, Mahwah, NJ: Lawrence Erlbaum, 3–33.

Lesh (1981): R. Lesh, Applied mathematical problem solving, *Educational Studies in Mathematics*, 12(2), 235–265.

Michalewicz, Fogel (2001): Z. Michalewicz, D. B. Fogel, *How to solve it: Modern Heuristics*. Springer: Germany.

Milinković (2014): J. Milinković, Matematičko modelovanje u nastavnim sistemima, *Inovacije u nastavi*, XXVII/2, Beograd: Učiteljski fakultet, 45–55.

Milinković (2016): J. Milinković, *Ogledi o učenju u nastavi matematike*, Beograd: Učiteljski fakultet.

Milinkovic (2023): J. Milinkovic, *Visualization as a tool in teaching: Pre-service teachers competences on visualization* (In print).

Milinkovic, Mihajlovic, Dejic (2019): J. Milinkovic, A. Mihajlovic, M. Dejic, Effective choices of representations in problem solving, *Eleventh Congress of the European Society for Research in Mathematics Education*, Netherlands: Utrecht University.

Novick, Bassok (2005): L. R. Novick, M. Bassok, Problem Solving, In: K. J. Holyoak, R. G. Morrison (Eds.), *The Cambridge Handbook of Thinking and Reasoning*, Cambridge: University Press, 321–350.

Obradović, Zeljić (2015): D. R. Obradović, M. Ž. Zeljić, Metode i strategije rešavanja tekstualnih zadataka u početnoj nastavi matematike, *Inovacije u nastavi*, 28(1), 69–81.

Polya (1957): G. Polya, *How to solve it*, Princeton University Press.

Presmeg (2006): N. Presmeg, Research on visualization in learning and teaching mathematics, In: A. Gutierrez, P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future*, Rotterdam: Sense Publishers, 205–236.

Richardson (2004): K. Richardson, *A design of useful implementation principles for the development, diffusion, and appropriation of knowledge in mathematics classrooms*, Unpublished doctoral dissertation, Purdue University.

Roubíček (2007): F. Roubíček, Some aspects of communication in the teaching of mathematics – the semiotic viewpoint, In: J. Novotná, H. Moraová (Eds). *Proceedings of SEMT 2007, Approaches to Teaching Mathematics at the Elementary Level*, August 19–24, 2007, Prague, the Czech Republic: Charles University, Faculty of Education, 225–232.

Rütten, Weskamp (2019): C. Rütten, S. Weskamp, Conjectures and justifications in building towers of cubes and cuboids – design of a combinatorial learning environment, In: J. Novotná, H. Moraová (Eds.), *Proceedings of SEMT 2019, Opportunities in Learning and Teaching Elementary Mathematics*, Prague, the Czech Republic: Charles University Faculty of Education, 344–354.

Singer, Pelczer, Voica (2011): F. M. Singer, I. Pelczer, C. Voica, Problem posing and modification as a criterion of mathematical creativity, In: M. Pytlak, T. Rowland, E. Swoboda (Eds.), *Proceedings of the 7h Conference of the European Society for Research in Mathematics Education (CERME 7)*, University of Rzeszów, Poland, 9–13.

Terwel, Van Oersa, Van Dijka, Vanden Eeden (2009): J. Terwel, B. Van Oersa, I. Van Dijka, P. Vanden Eeden, Are representations to be provided or generated in primary mathematics education? Effects on transfer, *Educational Research and Evaluation*, 15(1), 25–44.

Venuto, Hart (2017): N. Venuto, L. C. Hart, First graders write about mathematics: a teaching experiment, In: J. Novotná, H. Moraová (Eds), *Proceedings SEMT 2017*, *Equity and diversity in elementary mathematics education*, August 20–25, 2017, Prague, The Czech Republic: Charles University, Faculty of Education, 421–430.

Watters James, English, Mahoney (2000): J. Watters James, L. English, S. Mahoney, *Mathematical Modeling in the Elementary School*, Australia: Queensland University of Technology.

Wittman (2005): E. Wittman, *Mathematics as the Science of Patterns – A Guideline for Developing Mathematics Education from Early Childhood to Adulthood*, plenary lecture presented at the International Colloquium "Mathematical Learning from Early Childhood to Adulthood" organized by the Centre de Recherche sur l'Enseignement des Mathematiques in collaboration with the Institut de mathématique de l'Université de Mons-Hainaut, Mons/Belgium, July 7–9, 2005. http://irem.u-strasbg.fr/php/publi/annales/sommaires/11/WittmannA.pdf

Woodlief (2007). A. M. Woodlief, Changing the Paradigm: Hypertext and Interactive, *Student-Centered Learning*. Retrieved in June 2022 from http://horizon.unc.edu/.

Јасмина В. Милинковић

Универзитет у Београду, Учитељски факултет

Оливера С. Симић

Основна школа „Боривоје Ж. Милојевић", Крупањ

ИЗАЗОВИ РЕШАВАЊА ВИЗУЕЛНО ПРЕДСТАВЉЕНИХ МАТЕМАТИЧКИХ ПРОБЛЕМА

Резиме: Организација наставног процеса која омогућава стицање квалитетних и ефективних математичких знања примењљивих у различитим животним ситуацијама и која поставља основу за целоживотно учење заснива се на решавању проблема. Подучавање у решавању визуелно представљених проблема један је од циљева који доприноси овим општим циљевима. Основни циљ истраживања представљеног у овом раду јесте експериментално испитивање ефеката проблемске наставе у развоју способности математичког моделовања које укључује визуелно представљене проблеме. Испитује се ефикасност наставе визуелно представљених задатака у развијању вештина математичког моделовања у решавању једначина и неједначина у четвртом разреду основне школе. Дескриптивна метода коришћена је за анализу, обраду и интерпретацију резултата истраживања како би се испитале врсте грешака које ученици праве при раду са визуелно представљеним информацијама. Учесници експерименталног програма показали су виши ниво знања у решавању једноставних и сложених једначина и неједначина, као и у састављању текстова на основу задатих иконичких приказа једначина и неједначина, за разлику од ученика који нису били под утицајем експерименталног модела. С обзиром на позитивне ефекте на развој вештина математичког моделовања, визуелно представљена проблемска настава оправдава потребу за применом у настави математике на млађем школском узрасту.

Кључне речи: визуелизација, постављање проблема, решавање проблема, математичко моделовање, једначина и неједначина.

Marija M. Vorkapić
Mila S. Milošević
Olivera J. Đokić
University of Belgrade
Teacher Education Faculty
Department of Didactics of Mathematics

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YOUNG PUPILS' ABILITY TO SOLVE PERSPECTIVE-TAKING PROBLEMS[1]

Abstract: Spatial ability is an integral part of mathematics teaching and learning, but not every component of this ability has received enough research attention. In this paper, we focus on young pupils' ability to solve two types of imaginary perspective-taking (IPT) problems given in the form of a test (the paper–pencil test). The results show a difference in solving imaginary perspective-taking problems between preschoolers and second-grade pupils who took part in this research. Still, even the second-grade pupils have not fully developed this special spatial ability because they are slightly less successful in appearance IPT2 tasks than in visibility IPT1 tasks. We noticed individual differences in both age groups. In addition, the preschool sample from Serbia is equally successful as the children from the Netherlands and significantly better than the children from the Cyprus sample of the same age reported by Van den Heuvel-Panhuizen, Elia and Robitzsch (2015). The general conclusion and educational implication are that imaginary perspective-taking ability should be nurtured more in early school years.

Keywords: imaginary perspective-taking, visibility, appearance, spatial reasoning, young pupils, Serbia.

INTRODUCTION

Determining the position occupied by objects in space has an adaptive significance for all living beings. We look for a path to navigate, manipulate objects in our daily lives, or imagine situations in which we may find ourselves (Newcombe, Huttenlocher 1992; Sinclair, Bruce 2014). In addition, it is often useful to be able to predict the position that an object can occupy in space with respect to different viewpoints, i.e. perspective-taking (Van den Heuvel-Panhuizen, Elia, Robitzsch 2014, 2015). As spatial reasoning is a key cognitive ability illustrated by the above

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marija.vorkapic@uf.bg.ac.rs

examples and supported by scientific research (Davis, Spatial Reasoning Study Group 2015), this article examines the ability of children of preschool and younger school age in Serbia to take an imaginary perspective.

Modern mathematics curricula emphasize the need to start with 3D geometry at an early age (Van den Heuvel-Panhuizen, Elia, Robitzsch 2014), which is in line with Freudenthal's (1973) position on early geometry learning. Therefore, spatial ability is important for young children's learning, and it is valuable to gain as much insight as possible into how children develop this ability. The major component of spatial ability is imagining objects from different observer perspectives (Van den Heuvel-Panhuizen, Elia, Robitzsch 2014, 2015). We focus on this specific spatial competence of preschoolers and children of younger school age, i.e. on the competence of imaginary perspective taking (IPT), which means children's ability to take a certain point of view mentally and to be able to make conclusions about the positions of objects from an imaginary perspective spatial competence (IPT).

THEORETICAL BACKGROUND

We follow a new path in mathematics education research. In a comprehensive review of recent research within the CERME 11 thematic working group (TWG) on the early years of mathematics, POEM4 conference and ICME-13 monograph *Contemporary research and perspectives on early childhood mathematics education* we have identified three recurring themes: 1) early interventions and their effects, 2) facilitating factors for learning and development and 3) key mathematical concepts that can be observed in children (Björklund, Van den Heuvel-Panhuizen, Kullberg 2020). We follow the directions of Björklund, Van den Heuvel-Panhuizen and Kullberg (2020) for research on early mathematics teaching and learning to get a deeper insight into what mathematics means to young children and how the foundations can be laid for the domains of spatial and geometric thinking.

In the early years, there is still an insufficient focus on the development of geometry and spatial thinking (Sinclair, Bruce 2014; Uttal, Cohen 2012; Đokić 2018; Đokić, Boričić, Jelić 2022). The first indicator that this situation is changing for the better is extensive recent research that consistently shows a strong connection between children's spatial abilities and achievements in mathematics and science (Newcombe 2010), which increases the probability of achieving better achievements in STEM disciplines (science, technology, engineering, and math) (Young, Levine, Mix 2018; Wai, Lubinski, Benbow 2009). Secondly, a growing number of indicators that children come to school with developed informal spatial reasoning exist (Bryant 2008; Ishikawa, Newcombe 2021), which is often not supported in the continuation of mathematics education through curricula that require the development of numerical and algebraic ways of thinking.

COMPONENTS OF SPATIAL ABILITY

Spatial ability is considered an autonomous intellectual ability (Clements, Battista 1992). Factor analysis identified specific abilities that make up spatial ability: spatial orientation, mental rotation, and spatial visualization (1925–1979) (Mix, Cheng 2012: 200). Each of the listed components refers to specific requirements. Spatial orientation refers to the perception of the position and changes of objects in space, mental rotation to the mental manipulation of remembered objects with regard to the degree of rotation, and spatial visualization to understanding complex spatial patterns and imagined movements of objects in space. In addition to the specified specific components, a meta-analysis of definitions of spatial abilities (1975–2011) also mentions disembedding, spatial perception, and imaginary perspective-taking (adapted from Uttal et al. 2013: 355). Imaginary perspective-taking is the visualizing of an environment in its entirety from a different position.

Isolating and researching each of the mentioned specific spatial abilities is important because the analyses of individual differences show that people who are good in one component are not necessarily good in others (Newcombe, Uttal, Sauter 2013). Here is an example of the research that points to a distinction between mental rotation and perspective-taking. Although formally and logically equivalent, mental rotation and perspective-taking are psychologically very different constructs. Subsequent studies have confirmed neural differences, but also individual differences that separate these two spatial abilities (Newcombe, Huttenlocher 2006). Newcombe and Huttenlocher state that Piaget conceptualizes them as a part of different developmental lines and that they occur at different ages in children. Thus, it is easier to select an image that shows the appearance of an object or sequence after imagining it rotating on its axis than after imagining moving to take a different perspective on the same object or sequence. However, mental rotation is not always simpler than perspective-taking. When people ask themselves which object in a sequence would be in a certain position relative to them after the transformation (e.g. *What would be closest? What would be to the left?*), imaginary perspective-taking is easier than mental rotation.

There is a fairly extensive literature showing the relationship between math performance and specific spatial abilities: mental rotations, spatial visualization, and visuospatial working memory (e.g. Lubinski 2010; Newcombe 2010; Uttal et al. 2013). There is little to no evidence for other specific spatial abilities, such as imaginary perspective-taking, map reading, or model use (Mix, Cheng 2012). The literature is also geared towards older children and adolescents, leaving us with relatively little information on the relationship between spatial abilities and math performance in younger children.

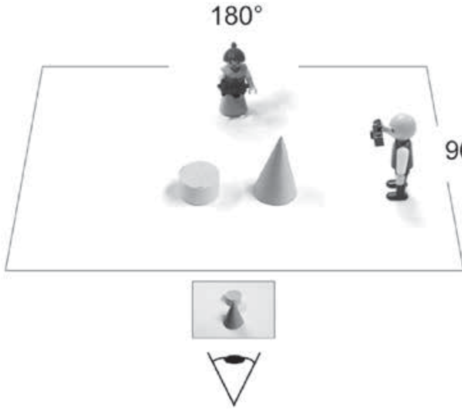
One of the few and earliest studies that singled out the imaginary perspective-taking as a separate, specific ability is the research of Guay and McDaniel (1977) who tested whether the ability to coordinate multiple points of view is related to achievement on the math subtest of the Iowa Test of Basic Skills. It is interesting that there was a significant correlation in the 5th, 6th, and 7th grades but not in the 2nd, 3rd, and 4th grades.

Although imaginary perspective-taking may not seem as obviously related to mathematics as other specific abilities, such as spatial visualization, there is a reason to suspect a connection. First, it seems likely that imaginary perspective-taking would be related to geometry, given the need to conceptualize the shapes of geometric objects from different angles and perspectives. A good perspective-taking ability may also reflect a level of "spatial flexibility" that would allow children to see equivalence in different situations, such as different solutions in algebra and proofs in geometry. Finally, the same processes that allow people to label and maintain separate viewpoints in working memory can also help them complete multi-step math problems, such as solving inequalities or adding fractions with different denominators. This is because keeping track of what is happening on both sides of the equal sign could involve the same processes needed to keep track of the multiple viewpoints (Mix, Cheng 2012; Uttal et al. 2013).

IMAGINARY PERSPECTIVE-TAKING

Emphasizing the importance of transdisciplinary research, Bruce et al. (2017) examined the lack of mutual understanding of the disciplines of mathematics education, psychology, mathematics, and neuroscience on the concept and development of spatial reasoning. To gain an insight into the existing connections between the disciplines, they conducted a network analysis that showed that taking a perspective is a representative case of overlapping psychological and mathematical education research. In psychology, imaginary perspective-taking is a cognitive construct that originated in the seminal work of Jean Piaget (Piaget, Inhelder 1948/1967). In Piaget's classic test, "The Three Mountains Task", a child is presented with a landscape scene and asked to describe it from other perspectives. The construct is often associated with egocentrism and consideration of others' points of view (Piaget 1932/1997). More recently, variations of the task have contributed to a better understanding of children's perspective-taking abilities. For example, Frick, Möhring and Newcombe (2014) designed a task involving a three-dimensional setting. Children were asked which of the two Playmobil figures took the photo shown above the picture of the eye (Figure 1). Their results show that egocentric errors occur mostly at the age of four and five and increase with the complexity of the request, but also decrease with age, i.e. the number of children who perform above the chance level increases at the age of five and six, while children at the age of seven and eight significantly reduce the number of errors, although even then individual differences are considerable.

Figure 1. Imaginary perspective-taking task (Frick, Möhring, Newcombe 2014: 3)



Earlier, Newcombe (1989) found that many studies rejected Piaget's age limitations, showing that children can overcome their egocentrism early in the preschool years. Thus, Newcombe and Huttenlocher (1992) provide evidence that even three-year-olds can solve perspective-taking problems by shifting from an egocentric to an allocentric frame of reference, i.e. children of preschool age can show the ability to take different perspectives of the object. Not all components of spatial abilities develop at the same age and particularly mental rotation appears a year or two later. However, variant forms remain severe at this age and are slow to develop, with good achievements not evident until the age of 10 (Newcombe, Huttenlocher 1992). Although imaginary perspective-taking is poorly researched within mathematics education (unlike psychological research), it becomes present in curricular outcomes, e.g. in three-dimensional structures. Bruce et al. (2017) report an outcome in the Ontario mathematics curriculum: build three-dimensional models using stacking cubes, according to a given isometric sketch or different views of the structures (top, side and front) (give an example problem: Given a top, side and front view of a complex structure, stack the body using the smallest possible number of cubes) (Ontario Ministry of Education 2005: 92). Although the given example is quite simple, it draws our attention to the presence of imaginary perspective-taking in many aspects of mathematics and in many activities such as drawing a structure, assembling and disassembling it, and navigation and mapping.

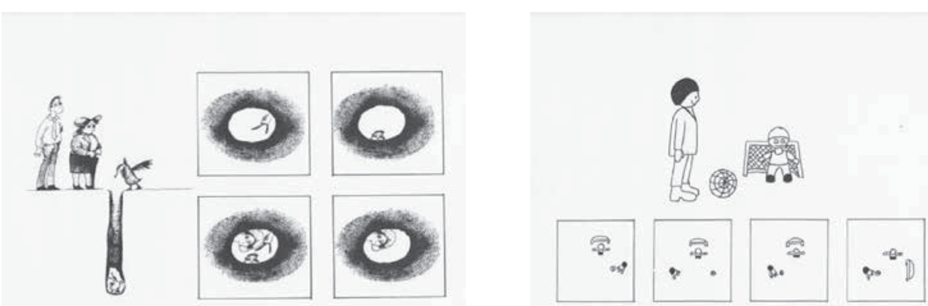
We know little about how spatial ability affects children's development. Does the relationship between spatial abilities and math performance change over time? Is the relationship more important at one point in development than at another? Does it involve children's specific abilities or do these abilities change? Before we can begin to ask *why* spatial abilities and math performance are related, we need to know a lot more about *how* they are related.

TWO TYPES OF IMAGINARY PERSPECTIVE-TAKING

Based on the previous research, Everett, Croft and Flavell (1981) proposed and confirmed the distinction of two specific perspective-taking abilities, distinguishing two levels of children's competencies. Level 1 competence concerns the visibility of objects, i.e. the ability to conclude which objects are visible and which are not from a certain point of view. Level 2 competence is related to the appearance of objects, that is, the ability to make a judgment about how the object looks from a certain point of view. Van den Heuvel-Panhuizen, Elia and Robitzsch (2015) in their research on imaginary perspective-taking (IPT) of preschoolers take the terms and talk about IPT type 1 (visibility) and IPT type 2 (appearance).

The aim of the study by Van den Heuvel-Panhuizen, Elia and Robitzsch (2015) was to gain a better insight into the IPT of preschoolers, especially the developed abilities of IPT type 1 (visibility) and IPT type 2 (appearance) and cross-cultural patterns in this competence including children from two countries, the Netherlands and Cyprus. Specifically, Van den Heuvel-Panhuizen, Elia and Robitzsch investigated the extent to which preschool children developed IPT type 1 and type 2 abilities, how these competencies are related, and whether IPT competency is related to children's age, mathematical abilities, and gender. The sample consisted of four and five-year-old children in the Netherlands (N = 334) and Cyprus (N = 304). Children's IPT competence was assessed with a paper-and-pencil test of different pictorial representations of objects that take perspective into account and require IPT type 1 or IPT type 2. Figure 2 shows two items from the test. The Duck item (instruction: "The duck has fallen into the hole. He looks up. What does he see?") is intended for measuring IPT type 1, while the Soccer item (instruction: "Two children are playing soccer. How do you see it if you look from above like a bird?") intended for measuring IPT type 2.

Figure 2. Two items: a) Duck item and b) Soccer item to measure IPT type 1 and type 2 respectively (Appendix)Figure 2



The results showed interesting common patterns for the two IPT types in both countries. Specifically, IPT type 2 items were significantly more difficult to solve than IPT type 1 items, and children's achievements on the first items imply achievements on the latter. Also, in both countries, IPT type 1 appeared to develop during preschool years. For IPT type 2 this was the case only in the Netherlands. There were no significant gender differences in IPT competencies among the preschoolers in the two countries. However, the relationship between children's IPT competence and mathematical abilities was not so clear.

Attention to spatial abilities is mentioned in the relevant papers of Mammana and Villani (1998), NCTM (2000) and NRC (2009), as well as abilities in specifying locations of objects (including interpreting relative positions in space) and using visualization (including creating mental images of geometric objects using spatial memory and spatial visualization, as well as recognition and representation of objects from different perspectives). Similar approaches can be found in curricula in England (Department of Education 2013), Australia (Board of Studies New South Wales 2012), the Netherlands (Van den Heuvel-Panhuizen, Buys 2008), and Cyprus (Cyprus Ministry of Education and Culture 2010). In Serbia, such approaches in mathematics curricula are not sufficiently recognized and, therefore, represent potentially unequal opportunities for learning and developing spatial ability and spatial reasoning ability (Pravilnik o planu nastave i učenja za prvi ciklus osnovnog obrazovanja i vaspitanja i programu nastave i učenja za prvi razred osnovnog obrazovanja i vaspitanja 2017; Pravilnik o programu nastave i učenja za drugi razred osnovnog obrazovanja i vaspitanja 2018; Pravilnik o programu nastave i učenja za treći razred osnovnog obrazovanja i vaspitanja 2019; Pravilnik o programu nastave i učenja za četvrti razred osnovnog obrazovanja i vaspitanja 2019). Here are the results of the analysis of the mentioned documents (mathematics curriculum) for the Space area in the Geometry domain:

1. In the first grade for seven-year-old children, outcome states that they are able to determine the mutual position of objects and beings and their positions in relation to the ground, as well as to notice and name the geometric shapes of objects in the immediate environment.
2. In the second and third grades, the outcomes that explicitly refer to the development of spatial thinking are not stated.
3. In the fourth grade, the outcome related to the recognition of the pictorial representation of the body viewed from different sides is stated.

At preschool age, children are focused on exploring the space around them, that is, in a real environment (Opšte osnove predškolskog programa 2006; Osnove programa predškolskog vaspitanja i obrazovanja 2018). The question arises as to how many educators put them in different situations that require positions of perception and description of different spatial situations.

From pre-school onwards, children should be provided with learning opportunities to further develop their spatial skills. This is an indicator that children in the continuity of mathematics education through curricula in Serbia are not clearly supported in the development of spatial reasoning, including the spatial ability of imaginary perspective-taking.

METHODOLOGY

The subject of this research is two types of imaginary perspective-taking as a special spatial component that develops from an early age. The research was conducted in May 2022. A total of 88 children, divided into two groups, participated in the study: preschool children (31) and primary second-grade pupils (57). All participants are from urban areas. This research aimed to examine the level of imaginary perspective-taking ability in children aged six to eight years old. In reference to this goal, research questions arise:

1. What is the general performance of the preschool children and second grade pupils on the imaginary perspective-taking test?
2. a) Are children in both age groups, preschool and primary school, equally successful on an imaginary perspective-taking test and parts of this test? b) Are the participants of a certain age group equally successful at two types of imaginary perspective-taking IPT1: *visibility*, or which objects are visible, and IPT2: *appearance,* or how an object or situation looks from a certain point of view.
3. What is the general performance of the children on individual tasks on the imaginary perspective-taking test?
4. Are the preschoolers from Serbia equally successful as their peers from the Netherlands and Cyprus on the imaginary perspective-taking test and the parts of the test?

To address the research questions a descriptive method and paper and pencil test were used. The imaginary perspective-taking test consists of 13 tasks, out of which seven tasks refer to IPT1, visibility, and six refer to IPT2, appearance. Each task had four possible answers and participants could reach a maximum of 13 points. The instrument was taken from the research of Van den Heuvel-Panhuizen, Elia and Robitzsch (2015) with little technical adaptations and as such it is presented in the Appendix. We calculated the reliability of IPT items by using Cronbach's alpha. The reliability of 13 items was $\alpha = 0.64$. The identified reliability is considered good, but below the frequently used minimal criterion of 0.70. However, given the heterogeneous nature of IPT, such low reliability can be expected (Cortina, 1993).

For data processing, we used SPSS 22. The general performances of the participants were analyzed using descriptive statistical measures. In order to examine if the test scores have normal distribution, we used the Shapiro–Wilk normality test. Intending to determine the statistical differences between the preschool age group and primary school age group in an imaginary perspective-taking test, we ran the Mann–Whitney U test. For determining the statistical differences in individual items between the groups we conducted Pearson's Chi-square test of homogeneity. Further, to examine if the participants are equally successful in both IPT types we used the paired sample t-test for the preschool group and Wilcoxon Signed Rank Test for the primary school group of pupils. By using One-sample t-test we wanted to compare Serbian preschool children's success with the success of the Netherlands and Cyprus children (Van den Heuvel-Panhuizen, Elia, Robitzsch 2015).

RESULTS AND DISCUSSION

The first research question refers to participants' general performance on the imaginary perspective-taking test. Children could reach the maximum of 13 points on the test. The results are shown in Table 1.

Table 1. General performance of the participants on the IPT test.

age group	*N*	*M*	*SE*	*SD*	*Mdn*	Range[a)]	Mod[a)]	Shapiro-Wilk
preschool	31	0.59	0.03	0.16	0.62	8	7 and 9	0.97*
second-grade	57	0.83	0.02	0.14	0.84	6	11	0.89**

*p = 0.41
**p = 0.00
[a)] calculated for number of points on the test

The results show the mean score or proportion of correct answers. The mean score of correct answers for preschool children is M = 0.59 (SD = 0.16), and for second-grade pupils it is M = 0.83 (SD = 0.14). Although the mean score of the second-grade participants indicates that the test was easy for them, the range of six points shows us that not all of the students developed IPT ability. Shapiro-Wilk normality tests show that preschool children's performance follows normal distribution, but that is not the situation in the group of the second-graders. This might indicate that the test is not suitable for students older than preschool age, but with further analysis we come to some conclusions and insights.

Further, we aimed to a) compare the preschool children's success with the primary school pupils' success on the IPT test and parts of the test; b) explore if participants of a certain age group are equally successful in both IPT types (IPT1, visibility and, IPT2, appearance).

From general performance results we have seen that older participants were expectedly better than the younger ones, so we wanted to explore with the Mann–Whitney U test if those differences are statistically significant. The rank test confirmed our doubt that the difference between the preschool children (Mdn = 0.62, n = 31) and the second-grade pupils (Mdn = 0.85; n = 57), U = 240, z = -5.67, p = 0.00, r = 0.6) is statistically significant and the calculated effect size is considered large. A significant outperformance was found on the parts of the test that is on IPT1 visibility items $U = 285.50$; $z = -5.39$, $p = 0.00$; $r = 0.6$ ($Mdn = 0.57$, $n = 31$ for preschoolers and $Mdn = 0.86$; $n = 57$ for second graders) and IPT2 appearance items $U = 409$; $z = -4.24$, $p = 0.00$; $r = 0.5$ ($Mdn = 0.50$; $n = 31$ for preschoolers and $Mdn = 0.83$; $n = 57$ for second graders).

Considering that older participants outperformed the younger ones, we were curious to explore how IPT ability develops within age groups according to two types of this ability. The paired sample t-test has shown that preschool children were equally successful in the IPT1 ($M = 0.60$, $SD = 0.23$) and in the IPT2 items ($M = 0.58$, $SD = 0.18$), $t(30) = 0.395$, $p = 0.696$. With school age participants' situation is slightly different. Wilcoxon Signed Rank Test indicates that second grade students are significantly better in IPT1 items ($Mdn = 0.86$) then in IPT2 items ($Mdn = 0.83$) $z = -2.670$, $p = 0.008$, $r = 0.25$. However, this difference is considered small.

Within two types of imaginary perspective-taking, we examined the participants' success in individual items. The first seven items are IPT1 visibility items and the rest of the items are IPT2 appearance items. The results are shown in Table 2.

Table 2. Participants' success on individual IPT items.

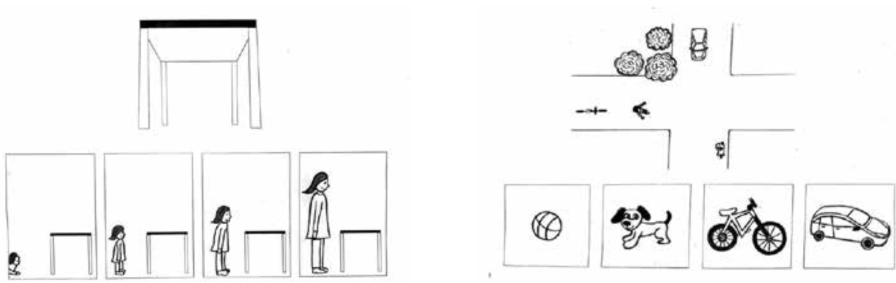
Task	Items	preschool age %	school age second grade %	χ^2 (1, N = 88)[a)]
1	IPT1 Umbrella	77.4	73.3	0.017
2	IPT1 Duck	77.4	87.7	0.915
3	IPT1 Crossing	25.8	78.9	21.505*
4	IPT1 Basket	54.8	98.2	23.774*
5	IPT1 Tower	48.4	93.0	20.270*
6	IPT1 Wall	80.6	98.2	6.262*
7	IPT1 Hole	54.8	78.9	4.508*
8	IPT2 Mouse	71.0	89.5	3.643
9	IPT2 Cucumber	77.4	73.7	0.017
10	IPT2 Fence	45.2	78.9	8.901*
11	IPT2 Soccer	80.6	77.2	0.011
12	IPT2 Table	29.0	77.2	17.484*
13	IPT2 Tree	45.2	73.7	5.881*

[a)] Yates correction was used for 2x2 tables.
* $p < 0.05$

The result of the Pearson's Chi-square of homogeneity indicates differences in success between the two age groups in the following tasks: Crossing, Basket, Tower, Wall, Hole (IPT1), Fence, Table, and Tree (IPT2). As the second-graders outperformed the younger participants in five of the mentioned items that are

IPT1 items, we can see, once again, that the second-graders are not as successful in IPT2 items as in IPT1 items. The preschoolers achieved better results in Umbrella, Cucumber, and Soccer tasks, but this is insignificant. We noticed that preschool participants are more successful in visibility items that are posing bird's or bottom view perspectives such as in Umbrella and Duck (77.4% success in both items), but less successful in visibility tasks that involve someone else's perspective such as in Crossing, Basket, Hole or Tower (25.8%, 54.8%, 54.8%, 48.4% respectively). In the latter items, we have an angled view, which might be slightly difficult for younger children. Similarly, in appearance items that include bird's perspective, such as item Mouse,(71%), the younger participants were remarkably successful. In this task, a front view of the mouse is drawn, and children have to choose between one of four drawings that shows the mouse from a bird's eye perspective (Mouse and other tasks are available in Appendix). The preschoolers have better performance in Mouse task than in Table (29%) which is also an appearance item (see Figure 3). As Figure 3 shows, children should imagine how each of the four girls sees the table and choose the correct answer. We have not noticed such regularity in the second-graders' performances. The older students outperformed the younger ones in the tasks that require some experience. For example, look at the Crossing item (Figure 3). Second-graders are, we assume, more independent in traffic than preschoolers. In this task the children have to choose what the boy sees while walking down the street. The second-grade pupils demonstrated a high performance in every item, but it seems that IPT1 Umbrella (73.3%), IPT2 Cucumber (73.7%) and Tree (73.7%) are items that are more challenging for them. In the Umbrella task, the girl is holding an umbrella, a flower, and a ball under it. Children should choose what is seen from a bird's eye perspective (the girl's head, the flower, the ball or the surface of the umbrella). In the cucumber task, children should choose the appropriate picture that represents the cross section of a cucumber according to the picture that shows a knife cutting the cucumber. In Tree task, a tree is shown upside down with a mouse hanging on a branch. Children should choose what the tree would look like in reality. We consider that the Tree item is more complex than other tasks because of the given answers that include mental rotation. An explanation for Cucumber task can

Figure 3. Two items: a) Table to measure IPT 2 and b) Crossing to measure IPT 1 (Appendix).



be that when we cut a cucumber, a cross-section shape is a circle, but perhaps it is not so in pupils' experience nowadays (chopping machines are used).

We have analyzed which incorrect answers the participants usually chose, and we have some interesting insights. First, we point out that in most of the tasks the preschoolers have chosen more incorrect answers than the second-graders. The most frequently chosen incorrect answer by the children of this age was the answer c) in item Table (see Appendix, IPT 2 – Table), where as many as 41.9% of children considered this to be the correct answer. Answer c) in item Table might be the most natural position for children of five to six years, but they did not consider the instruction that points out a concrete picture of the table. Among the second-grade pupils, the common incorrect answers appeared in two tasks: Cucumber (answer c), 21.1%) (see Appendix, IPT 2 – Cucumber), and Tree (answer d), 21.1%), (see Appendix, IPT 2 – Tree). The incorrect answer c) in the Cucumber item shows an intersection as the correct answer (instead of answer b), so the pupils who were wrong might have been rushing or did not understand the instruction. The incorrect answer b) in the Tree item is similar to the correct answer d), but the animal on the tree is on the wrong side of the tree. A greater number of errors by the second-graders is also visible in the Umbrella task, where more pupils gave an incorrect answer b) (15.8%) and c) (10.5%) (see Appendix, IPT 1 – Umbrella, correct d) and in the Soccer task (answer b), 10.5%) (see Appendix, IPT 2 – Soccer, correct c). Based on the quantitative data, we can see that the children had the lowest number of incorrect answers in the Wall task in which a wall is placed between two children, and the question is in which situation the children can see each other depending on the height of the wall between them (see Appendix). This may imply that the situation presented in this task is intuitively and experientially closest to them. However, based on the data, we cannot come to a single conclusion as to why children choose a certain situation, that is, a picture as one of the incorrect answers. For that kind of conclusion, an interview would be a more suitable technique.

Finally, we wanted to compare the success of the preschool children in Serbia to the success of the children from Cyprus and the Netherlands from existing research (Van den Heuvel-Panhuizen, Elia, Robitzsch 2015). The one-sample t-test showed that the preschool children from our sample showed similar scores on the imaginary perspective-taking test as the children from the Netherlands (age = 5.69), $M = 0.59$, $SD = 0.16$, $t(30) = -0.327$, $p = 0.746$ (the tested value from the cited research is 0.60, that is, an average of 60% success on the test). However, compared to the children from Cyprus (age = 5.61), the preschool children from Serbia have a significantly higher score on the imaginary perspective-taking test $M = 0.59$, $SD = 0.16$, $t(30) = 3.833$, $p = 0.001$ (the tested value from the cited research is 0.48, i.e. 48% success rate on the test). We further compared the performance on the parts of the test, namely, on the tasks of imaginary perspective-taking type 1 and imaginary perspective-taking type 2 between the countries. The children from the Netherlands are significantly better on the IPT1 ($M = 0.74$) compared to the preschool children from our sample ($M = 0.60$, $SD = 0.23$), $t(30) = -3.432$, $p = 0.002$, but the children from Serbia are significantly more successful on IPT2 tasks ($M = 0.58$, $SD = 0.18$), $t(30) = 3.800$, $p = 0.001$ compared to the children from the Netherlands ($M = 0.46$). There is no difference in performance on IPT1 items between the children from Cyprus ($M = 0.61$) and children from our sample ($M = 0.60$, $SD = 0.23$), $t(30) = -0.266$, $p = 0.792$, but there is on IPT2 items in favor of the children from Serbia ($M = 0.58$, $SD = 0.18$), $t(30) = 7.580$, $p = 0.00$ ($M = 0.34$ mean score of children from Cyprus).

CONCLUSION

The results of the research on children's success in perspective-taking tasks show that second-grade students are more successful than preschool children, and we can conclude that as children grow older, their ability to take a perspective in a two-dimensional environment improves. The children of the second grade were more successful in relation to the preschool children and in relation to both types of perspective-taking abilities (IPT1 and IPT2), with deviations on specific perspective-taking tasks of type 1 (Umbrella) and tasks of type 2 (Cucumber and Soccer) where the preschoolers were more successful. This can be explained by the fact that the younger participants, often carried in the arms of their parents, had the opportunity to perceive objects from a bird's eye view and were extremely good at tasks of this type. On the other hand, upon enrolling at primary school, this ability was not further developed and nurtured, as evidenced in the mathematics curriculum (Pravilnik o planu nastave i učenja za prvi ciklus osnovnog obrazovanja i vaspitanja i programu nastave i učenja za prvi razred osnovnog obrazovanja i vaspitanja 2017; Pravilnik o programu nastave i učenja za drugi razred osnovnog obrazovanja i vaspitanja 2018), so the second-grade students have somewhat lower scores. Observing only a specific type of perspective-taking ability, we came to a conclusion that preschool children are equally successful in both types of perspective-taking, while second-grade pupils are somewhat more successful on visibility tasks (IPT1) compared to appearance tasks (IPT2). The analysis of the most frequently chosen incorrect answers showed us that children of preschool age and second grade have similar misconceptions, that is, difficulties when they are put in situations to determine whether an object is visible or how it looks from another perspective. However, these quantitative indicators cannot shed light on why children think that a certain answer, i.e. a picture, is more appropriate to the given situation. Some future research may address this issue. Since in the research we relied on the existing research instrument (Van den Heuvel-Panhuizen, Elia, Robitzsch 2015), we were also interested in the position of the preschool children from Serbia in relation to the children of the same age in Cyprus and the Netherlands. The results of our research showed that Serbian children from our sample are significantly better at solving perspective-taking tasks than the children from Cyprus sample, and equally successful as the children from the Netherlands sample. More precisely, when it comes to the types of perspective-taking abilities, the children from our sample are significantly more successful in the tasks referring to how something looks (IPT2), compared to the children from the samples from both countries, but less successful in the tasks of whether something can be seen (IPT1) compared to Dutch children and equally successful as Cyprus children. However, we emphasize that the sample in our research is small, and that further analyses of imaginary perspective-taking abilities should be conducted on a larger sample (both rural and urban areas), and the conclusions of such analyses can be generalized for a wider population.

Finally, we will refer to the instrument we used in the research itself, taken from the mentioned research (Van den Heuvel-Panhuizen, Elia, Robitzsch 2015). As noted, the test itself challenged preschoolers, while it was easy for the second-graders. In addition, we noticed certain shortcomings in certain tasks. For example, the Fence task did not send a clear message about which answer was correct since one of the first two answers offered could be considered the correct answer. Also, with the Tower task, it is not entirely clear in which direction the girl is looking, as well as whether certain objects can create a "distraction" or obscure how visible they are from a certain position. Furthermore, the Tree task also did not clearly depict the appearance of the object under consideration in the picture. Certain technical changes were made in the mentioned tasks. During the realization of the research, and based on the questions asked by the pupils of the second grade, we also observed that the task involving streets was unclear to them, namely, that based on the picture, they could not see in which direction the child in the picture was moving.

The aforementioned research results indicate that children of preschool and primary school age need to be placed in situations where they will have an opportunity to consider the visibility or appearance of a certain object from another perspective, especially as there are few such requirements in the preschool and primary school curricula, and there is a growing need to point out the importance of geometry and space (Sinclair, Bruce 2014; Uttal, Cohen 2012; Đokić 2018; Đokić, Boričić, Jelić 2022). In addition, we emphasize the need to examine in the future children's perspective-taking ability at other ages with an adequate instrument (such as in Frick Möhring, Newcombe 2014), as well as in other settings.

REFERENCES

Björklund, van den Heuvel-Panhuizen, Kullberg (2020): C. Björklund, M. van den Heuvel-Panhuizen, A. Kullberg, Research on early childhood mathematics teaching and learning, *ZDM Mathematics Education*, 52/4, Germany: Springer Verlag, 607–619.

Board of Studies New South Wales (2012): *NSW syllabus for the Australian curriculum Mathematics K–10 syllabus*. Sydney: Author.

Bruce, Davis, Sinclair, McGarvey, Hallowell, Drefs, Francis, Hawes, Moss, Mulligan, Okamoto, Whiteley, Woolcott (2017): C. D. Bruce, B. Davis, N. Sinclair, L. McGarvey, D. Hallowell, M. Drefs, K. Francis, Z. Hawes, J. Moss, J. Mulligan, Y. Okamoto, W. Whiteley, G. Woolcott, Understanding gaps in research networks: using "spatial reasoning" as a window into the importance of networked educational research, *Educational Studies in Mathematics*, 95/2, Netherlands: Springer Netherlands, 143–161.

Bryant (2008): P. Bryant, Paper 5: Understanding spaces and its representation in mathematics, In: T. Nunez, P. Bryant, A. Watson (Eds.), *Key understanding in mathematics learning: A report to the Nuffield Foundation*, London: Nuffield Foundation.

Clements, Battista (1992): D. H. Clements, M. T. Battista, Geometry and spatial reasoning, In: D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, Stuttgart: Macmillan, 420–464.

Cortina (1993): J. M. Cortina, What is coefficient alpha? An examination of theory and applications, *Journal of Applied Psychology*, 78/1, USA: American Psychological Association, 98–104.

Cyprus Ministry of Education and Culture (2010): *Program of studies in mathematics*, Nicosia: Cyprus Ministry of Education and Culture.

Davis, Spatial Reasoning Study Group (Eds.) (2015): B. Davis, Spatial Reasoning Study Group (Eds.), *Spatial reasoning in the early years. Principles, assertions, and speculations*, United Kingdom: Routledge.

Department for Education (2013): *Mathematics programmes of study: Key stages 1 and 2. National curriculum in England*, England: Department for Education.

Đokić (2018): O. J. Đokić, Research perspectives of initial geometry education, *Journal of Pedagogical Research*, 2/2, Turkey, 102–111.

Đokić, Boričić, Jelić (2022): O. J. Đokić, M. M. D. Boričić, M. S. Jelić, Comparing ICT with Physical Manipulative Supported Learning of 3D Geometry in Elementary School, *Journal of Educational Computing Research*, 59/8, USA: SAGE Publications Inc., 1623–1654.

Everett, Croft, Flavell (1981): B. A. Everett, K. Croft, E. R. Flavell, Young children's knowledge about visual perception: Further evidence for the Level 1 – Level 2 distinction, *Developmental Psychology*, 17/1, USA: American Psychological Association, 99–103.

Freudenthal (1973): H. Freudenthal, *Mathematics as an educational task*, Dordrecht, The Netherlands: D. Reidel Publishing Company.

Frick, Möhring, Newcombe (2014): A. Frick, W. Möhring, N. S. Newcombe, Picturing perspectives: development of perspective-taking abilities in 4- to 8-year-olds, *Frontiers in Psychology*, 5/386, Lausanne, Switzerland: Frontiers Media.

Guay, McDaniel (1977): R. B. Guay, E. D. McDaniel, The relationship between mathematics achievement and spatial abilities among elementary school children, *Journal for Research in Mathematics Education*, 8/3, USA: National Council of Teachers of Mathematics, 210–215.

Ishikawa, Newcombe (2021): T. Ishikawa, N. S. Newcombe, Why spatial is special in education, learning, and everyday activities, *Cognitive Research: Principles and Implications*, 6/20, USA: Springer Open.

Lubinski (2010): D. Lubinski, Spatial ability and STEM: A sleeping giant for talent identification and development, *Personality and Individual Differences*, 49/4, Amsterdam, Netherlands: Elsevier, 344–351.

Mammana, Villani (Eds.) (1998): C. Mammana, V. Villani, *Perspectives on the teaching of geometry for the 21st century – An ICMI Study*, Amsterdam: Kluwer Academic Publishers.

Mix, Cheng, (2012): K. S. Mix, Y. L. Cheng, The Relation Between Space and Math: Developmental and Educational Implications, In: J. B. Benson (Ed.), *Advances in child development and behavior*, 42, Amsterdam: Elsevier Inc.,197–243.

National Council of Teachers of Mathematics (NCTM) (2000): *Principles and standards for school mathematics*, Reston, VA: National Council of Teachers of Mathematics.

National Research Council (NRC) (2009): *Mathematics learning in early childhood: Paths toward excellence and equity*, USA: The National Academies Press.

Newcombe (1989): N. Newcombe, The development of spatial perspective taking, In: H. W. Reese (Ed.), *Advances in child development and behavior*, 22, New York: Academic Press, 203–247.

Newcombe, Huttenlocher (1992): N. Newcombe, J. Huttenlocher, Children's early ability to solve perspective-taking problems, *Developmental Psychology*, 28/4, USA: American Psychological Association, 635–643.

Newcombe, Huttenlocher (2006): N. S. Newcombe, J. Huttenlocher, Development of Spatial Cognition, In: D. Kuhn, R. Siegler (Eds.), *Handbook of Child Psychology* (6th ed.), 2, New Jersey: John Wiley & Sons, Inc., 734–776.

Newcombe (2010): N. Newcombe, Picture this: Increasing math and science learning by improving spatial thinking, *American Educator*, 34/2, USA: American Federation of Teachers, 29–35.

Newcombe, Uttal, Sauter (2013), N. S. Newcombe, D. H. Uttal, M. Sauter, Spatial development, In: P. Zelazo (Ed.), *Oxford handbook of developmental psychology*, 1, New York: Oxford University Press, 564–590.

Ontario Ministry of Education (2005): *Grade 6 mathematics curriculum*, Toronto: Ontario Education Ministry.

Pravilnik o opštim osnovama predškolskog programa (2006): *Prosvetni glasnik*, br. 14.

Pravilnik o osnovama programa predškolskog vaspitanja i obrazovanja (2018): *Prosvetni glasnik*, br.16.

Pravilnik o planu nastave i učenja za prvi ciklus osnovnog obrazovanja i vaspitanja i programu nastave i učenja za prvi razred osnovnog obrazovanja i vaspitanja (2017): *Prosvetni glasnik*, br. 10.

Pravilnik o programu nastave i učenja za drugi razred osnovnog obrazovanja i vaspitanja (2018): *Prosvetni glasnik*, br. 16.

Pravilnik o programu nastave i učenja za treći razred osnovnog obrazovanja i vaspitanja (2019): *Prosvetni glasnik*, br. 5.

Pravilnik o programu nastave i učenja za četvrti razred osnovnog obrazovanja i vaspitanja (2019): *Prosvetni glasnik*, br. 11.

Sinclair, Bruce (2014): N. Sinclair, C. D. Bruce, Spatial reasoning for young learners, In: P. Liljedahl, C. Nicol, S.Oesterle, D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA*, 36/1, Vancouver, Canada: PME, 173–203.

Uttal, Cohen (2012): D. H. Uttal, C. A. Cohen, Spatial thinking and STEM education: When, why and how?, In: B. Ross (Ed.), *Psychology of learning and motivation*, 57, New York: Academic Press, 148–178.

Uttal, Meadow, Tipton, Hand, Alden, Warren, Newcombe (2013): D. H. Uttal, N. G. Meadow, E. Tipton, L. L. Hand, A. R. Alden, C. Warren, N. S. Newcombe, The malleability of spatial skills: A meta-analysis of training studies, *Psychological Bulletin*, 139/2, USA: American Psychological Association, 352–402.

Van den Heuvel-Panhuizen, Buys (Eds.) (2008): M. Van den Heuvel-Panhuizen, K. Buys, *Young children learn measurement and geometry*, Rotterdam: Sense.

Van den Heuvel-Panhuizen, Elia, Robitzsch (2014): M. Van den Heuvel-Panhuizen, I. Elia, A. Robitzsch, Paper 6: Kindergarteners' abilities in perspective taking, In: P. Liljedahl, C. Nicol, S. Oesterle, D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA*, 36/1, Vancouver, Canada: PME, 184-185.

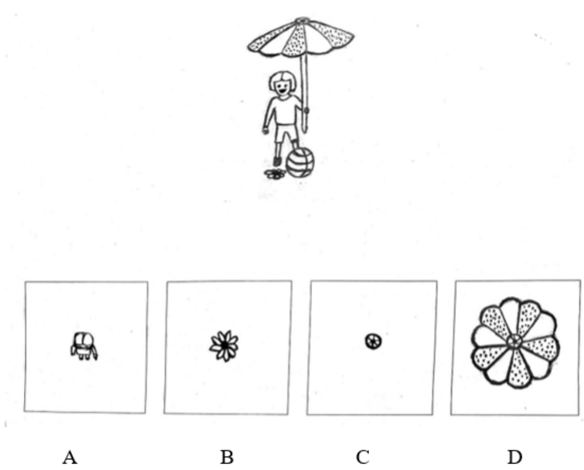
Van den Heuvel-Panhuizen, Elia, Robitzsch (2015): M. Van den Heuvel-Panhuizen, I. Elia, A. Robitzsch, Kindergartners' performance in two types of imaginary perspective-taking, *ZDM Mathematics Education*, 47/3, Germany: Springer Verlag, 345–362.

Wai, Lubinski, Benbow (2009): J. Wai, D. Lubinski, C. P. Benbow, Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance, *Journal of Educational Psychology*, 101/4, USA: American Psychological Association, 817–835.

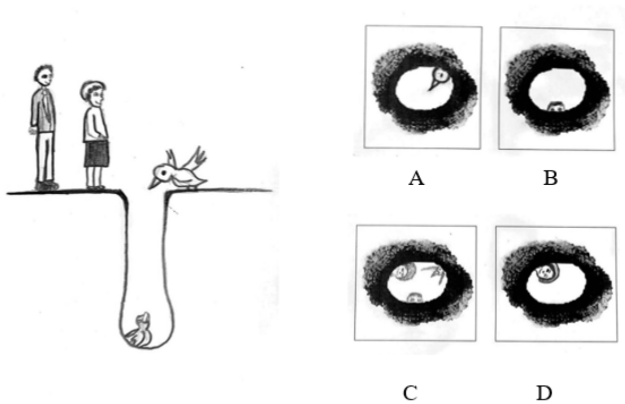
Young, Levine, Mix (2018): C. J. Young, S. C. Levine, K. S. Mix, The Connection between Spatial and Mathematical Ability across Development, *Frontiers in Psychology*, 9/755, Lausanne, Switzerland: Frontiers.

APPENDIX

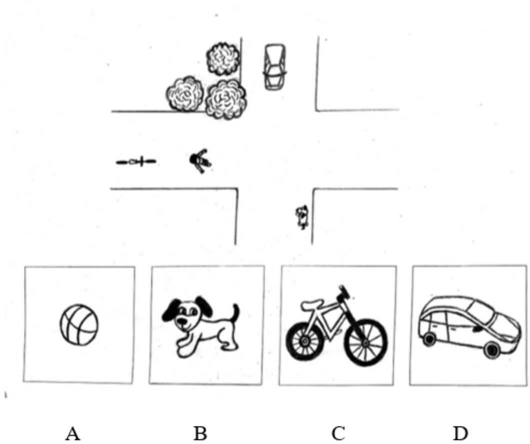
IPT1 – UMBRELLA



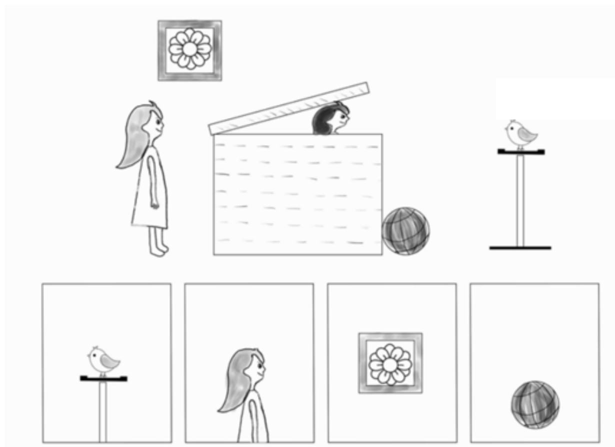
IPT1 – DUCK



IPT – CROSSING

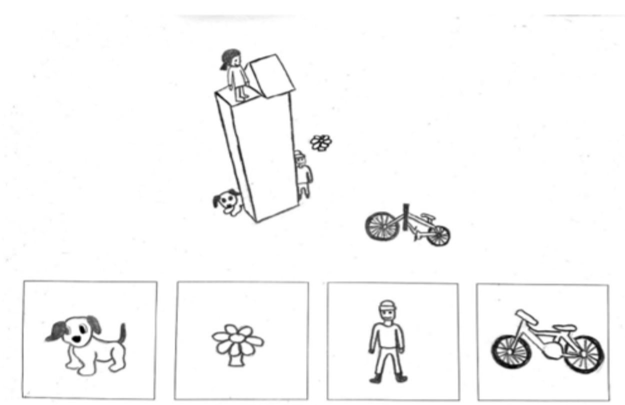


IPT 1 – BASKET



A B C D

IPT 1 – TOWER



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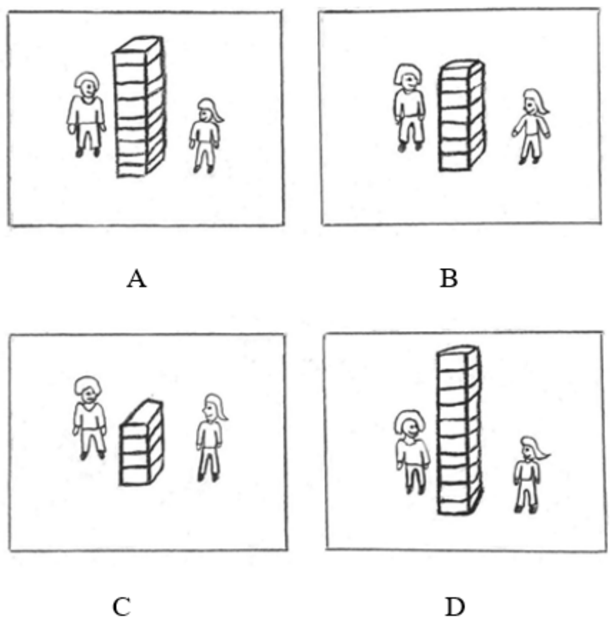
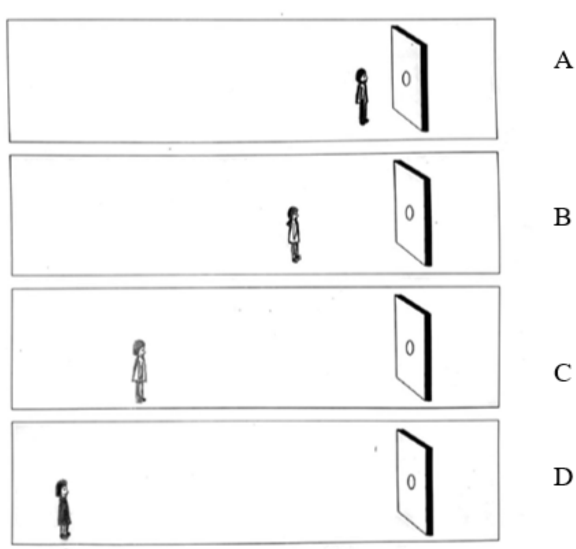
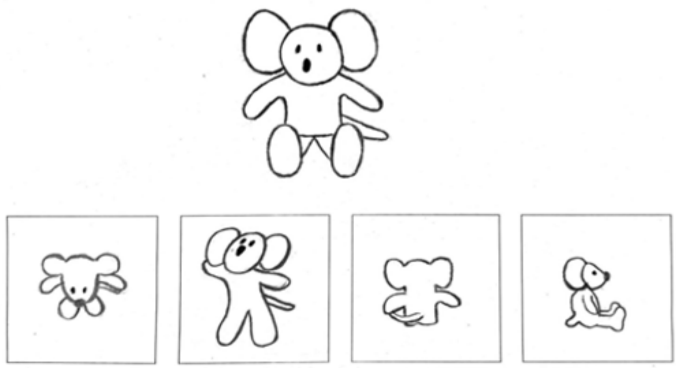
IPT 1 – WALL



IPT 1 – HOLE

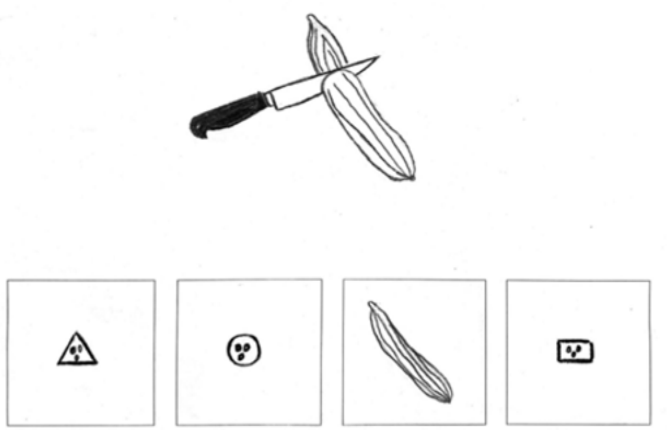


IPT 1 – MOUSE



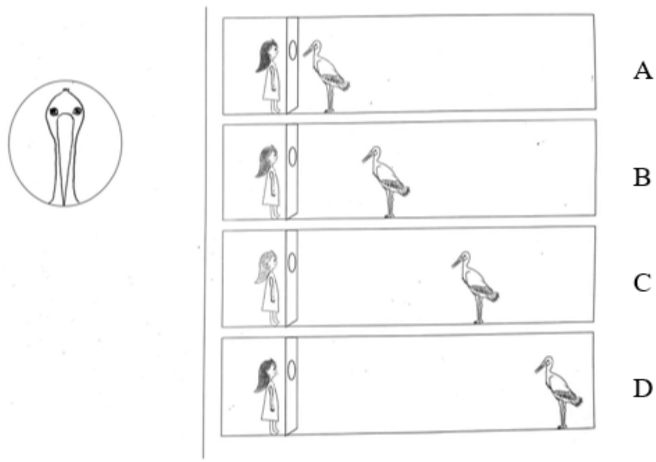
A B C D

IPT 2 – CUCUMBER

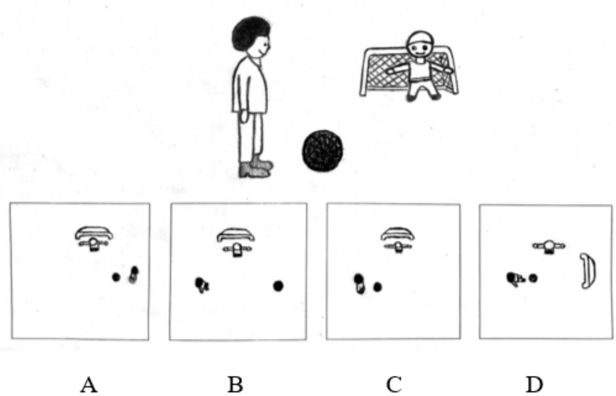


A B C D

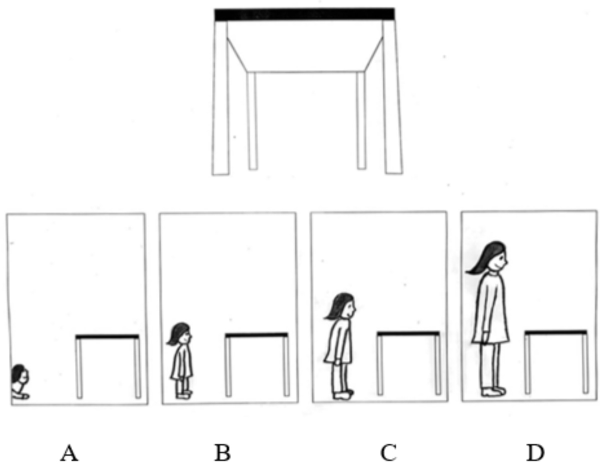
IPT 2 – FENCE



IPT 2 – SOCCER



IPT 2 – TABLE



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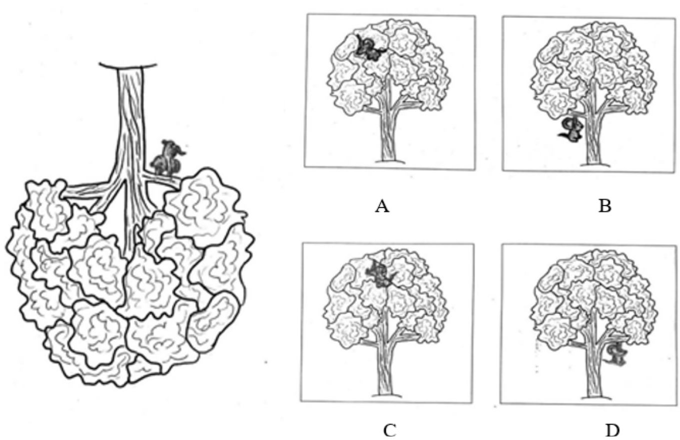

Марија М. Воркапић
Мила С. Милошевић
Оливера Ј. Ђокић
Универзитет у Београду
Учитељски факултет
Катедра за методику математике

СПОСОБНОСТ ЗАУЗИМАЊА ПЕРСПЕКТИВЕ КОД ДЕЦЕ ПРЕДШКОЛСКОГ И МЛАЂЕГ ШКОЛСКОГ УЗРАСТА

Резиме: Иако просторна способност представља битан елемент процеса учења и поучавања у математици, није свака компонента ове способности добила довољно истраживачке пажње. У овом раду се фокусирамо на способност деце предшколског узраста и ученика другог разреда основне школе при решавању две врсте проблема са заузимањем перспективе датих у облику теста (тест папир–оловка). Резултати показују разлику у решавању проблема сагледавањем заузимања перспективе између деце предшколског узраста и ученика другог разреда који чине узорак у овом истраживању. Осим тога, резултати показују да ученици другог разреда нису у потпуности развили ову посебну просторну способност, будући да су нешто мање успешни у задацима изгледа него задацима видљивости (видљивост и изглед представљају посебне способности заузимања перспективе). Приметили смо индивидуалне разлике у обе старосне групе. Поред тога, узорак деце предшколског узраста из Србије подједнако је успешан као деца из Холандије и значајно је бољи од деце са Кипра на узорку истог узраста који наводе Ван ден Хувел-Панхуизен, Елиа и Робич (Van den Heuvel-Panhuizen, Elia and Robitzsch 2015). Општи закључак и образовна импликација је да способност заузимања перспективе треба даље неговати у раном образовању и код деце млађег школског узраста.

Кључне речи: заузимање перспективе, видљивост, изглед, просторно резоновање, предшколска деца и деца млађег школског узраста, Србија.

Nenad S. Milinković
Sanja M. Maričić
University of Kragujevac
Faculty of Education in Užice
Bojan D. Lazić
University of Novi Sad
Faculty of Education in Sombor

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THE PROBLEM OF (MIS)UNDERSTANDING THE EQUALS SIGN IN JUNIOR GRADES OF PRIMARY SCHOOL

Abstract. Correct understanding of the equals sign is the key to understanding arithmetic, and a fundamental concept important for learning other areas of mathematics. Research around the world repeatedly mentions problems with correctly understanding the equals sign, emphasising the limited view of the equals sign as a command ”to calculate“ among students. The goal of the research was to identify the development of the concept of equality in line with the operationalised levels (operational, relational, and relational in the context of real-life problem solving) and determine differences in understanding between students of the second (N = 190) and the fourth (N = 210) grade of primary school. The research was carried out using the testing technique. The research results show that students do not possess sufficiently developed relational understanding of the equals sign and that operational understanding prevails. Students of the fourth grade demonstrated better understanding of the equals sign at all levels of understanding than the second graders.

Keywords: equals sign, equivalence, operational understanding, relational understanding, mathematics, mathematics education.

INTRODUCTION

The equals sign – a fundamental concept and symbol in mathematics. It is a concept formed in the earliest days of mathematics education, simultaneously with the formation of the concept of natural numbers. However, “equality is a central – but sorely neglected – concept in mathematics education” (Parslow-Williams, Cockburn 2008: 35). From the aspect of mathematical reasoning, the concept of equality involves at least three components: (a) understanding the equality of two values; (b) understanding the equals sign as the symbol of a relationship; and (c) the idea that there are two sides to the equality (Rittle-Johnson, Alibali 1999). All these components are critical to mathematics and problem solving but are often neglected in mathematics education. It is only when students encounter more abstract

milinkovic.nenad84@gmail.com

mathematical content that the need to understand this sign as a symbol representing equivalence arises.

Developing the correct understanding of this concept is often considered an easy task in mathematics education. The equals sign as the symbol of equivalence is a concept that, in a mathematical sense, represents the duality between the concept and the process (Alexandrou-Leonidou, Philippou 2007). If we wanted to define the concept of the equals sign (=) more accurately, we would say that it is a mathematical symbol that expresses equality between variables, constants, or other mathematical expressions. Research studies recognise two basic categories that explain how students use the equals sign, and what said sign essentially represents for them in mathematical equalities:

1) in operational sense, the concept of the equals sign stands for "find the total" or "the solution is";

2) in relational sense, the concept of the equals sign means that two expressions on the opposite sides of this sign express the same quantity/value (Kieran 1981; Knuth et al. 2006; McNeil et al. 2006).

This classification is based on the fundamental perception of the equals sign as an operation or computation on the one hand, and a relationship that expresses equivalence on the other hand. Considering this dual meaning that the concept of equivalence and the sign that expresses it share, it is possible to recognise their different understanding in algebra and arithmetic. A large body of research indicates that the primary source of difficulties that prevent students from correctly understanding the equals sign lies in their previous experience with it (Baroody, Ginsburg 1983; Carpenter, Franke, Levi 2003; Falkner et al. 1999; McNeil 2007, 2008).

Students are first introduced to this symbol in arithmetic classes where the equals sign is used in different forms, which may cause students to acquire an erroneous conceptualisation of it (Kieran 1981). Arithmetic equalities in which the expression is always on the left side of the equals sign often lead students to perceive the equals sign as an instruction that means "calculate" or "find the solution" (Baroody, Ginsburg 1983; Behr, Erlwanger, Nichols 1980; Cobb 1987; Ilić, Zeljić 2017; Kieran 1981, 1989). As a result, students begin to perceive the equals sign as an operation, interpreting it as the command for arithmetic calculations, i.e., they "see '=' as an instruction to complete an operation" (Parslow-Williams, Cockburn 2008: 36).

If we take the equality $12 + 5 =$ ____, for example, the student's first instinct will be to calculate the sum of *12* and *5*. Based on previous experience in arithmetic, students tend to always see the expression and equality in the same way, so that the expected result of the equality above will be *17*. If we consider the student's previous behaviour in encountering the expression and present them with the following equality $12 + 5 =$ ____ $+ 2$, students will struggle to understand the equivalence between the left and the right side of the equality. In this case, they will focus on calculating the sum on the left side of the equality, and, instead of arriving at the correct solution, i.e., number *15*, try to calculate the result of the expression on the left side of the equality $12 + 5 = 17 + 2$, which is incorrect. Through arithmetic content, students acquire the habit of perceiving the left side of the equality as the side where the instructions for the operation are defined, whereas the right side remains exclusively for expressing the results.

Research shows that problems can also be observed in later stages of mathematics education, whereby secondary school students have more difficulties when interpreting the equals sign in "non-standard" expressions (e.g. $3 + 4 = 5 + 2$ and $7 = 7$), than in expressions they are accustomed to (e.g. $3 + 4 = 7$) (McNeil et al. 2010). The roots of this mindset lie in the students' habit to calculate the result of the expression without understanding equality as a whole or identifying relationships between parts of the mathematical expression. Various studies conducted in the USA shows that a staggeringly high percentage of students (about 80%) between the ages of 7 and 11 are unsuccessful in solving problems designed to test their comprehension of mathematical equivalence (Alibali 1999; Baroody, Ginsburg 1983; Cobb 1987; Kieran 1981; McNeil 2007; RittleJohnson, Alibali 1999).

The first arithmetic expressions that students encounter have a huge bearing on the development of their perception and structural understanding of equality and mathematical expressions. Students who correctly understand the equals sign do not view the arithmetic problem as a signal to perform a specific operation, but instead learn to identify the relationship expressed in the equality before calculating the result (Jacobs et al. 2007). Dabić Boričić and Zeljić notice that if expressions are understood "as processes (calculating the value of expressions), and not as objects with a meaning of their own, students will understand algebraic expressions as evaluation procedures, instead of mental entities that can be manipulated" (Dabić Boričić, Zeljić 2021: 31).

The solution to this problem lies in the reshaping of our approach to learning such content. Thus, in situations where students solve problems that involve expressions with addition and subtraction, for example $\_\_\_\_ = 4 + 3$, the equals sign should be replaced with words that indicate equivalence: "is equal to", "two quantities are equal", "something is equivalent to something else", etc. Such examples can help expand the meaning of the equals sign as a concept, shifting it from operational to relational understanding. In the first case, student activity related to mathematical expressions is aimed at calculations, i.e., determining their result/value. Understanding the mathematical expression as an object, on the other hand, refers to understanding its structure as a whole that can exist on its own. Only when the student is able to understand a mathematical expression as an independent object can they reach structural understanding and deeper understanding of the expression, and thus master the concept of equality (Milinković, Maričić, Đokić 2022). Some authors recommend emphasising the link between the different meanings of the equals sign in teaching, especially between the meaning of the symbol, action, and numerical equivalence in order to present numerical equalities in an integrated manner (Molina, Castro, Castro 2009).

The root of all problems with understanding the equals sign lies in the deeper understanding of this concept and developing an understanding of this concept at the relational, instead of just the operational level. In their research, Rittle-Johnson et al. (2011) identified four levels of understanding of the concept of equality where "knowledge levels differ primarily in the types of equations with which students are successful, starting with equations in an operations–equals–answer structure, then incorporating equations with operations on the right or no operations, and finally incorporating equations with operations on both sides" (Rittle-Johnson et al. 2011: 3) (Table 1).

Table 1. Construct Map for Mathematical Equivalence Knowledge (Rittle-Johnson et al. 2011: 3)

Level	Description
Level 4: Comparative relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing that performing the same operations on both sides maintains equivalence. Recognize relational definition of equal sign as the best definition.
Level 3: Basic relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recognize and generate a relational definition of the equal sign.
Level 2: Flexible operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign.
Level 1: Rigid operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.

These levels should not be viewed as separate and unrelated stages, i.e., there is no clear boundary that excludes mutual ties, and students can develop different interpretations of the equals sign at the same time (Jones et al. 2012). This scaled operationalisation greatly facilitates the understanding of the concept of equality.

Based on the considerations regarding the understanding of the equals sign mentioned above and taking into account the research that operationalises levels of understanding of the equals sign (Kieran 1981; Knuth et al. 2006; McNeil et al. 2006; Rittle-Johnson et al. 2011; McAuliffe, Tambara, Simsek 2020), and finally, looking at the outcomes of mathematics education, we can distinguish between four levels of understanding of the equals sign (Table 2).

Table 2. Levels of understanding of the equals sign

Level of understanding	Expected outcomes
Level 4: Real relational	Student understands equivalence in real-world context problems.
Level 3: Complex relational	Student understands equivalence in complex equalities that feature multiple equals signs.
Level 2: Basic relational	Student understands the equals sign as a symbol of equivalence in equalities that feature expressions on both sides of the equality. Student uses relational thinking and understands equivalence in simple equalities.
Level 1: Operational	Student understands the equals sign as a command “to calculate”. Student understands simple equalities that feature expressions on both sides of the equals sign.

In the operationalisation above, the relational level of understanding involves three sublevels: basic relational, complex relational, and actual relational. The lowest sublevel of relational understanding is the understanding of equivalence in situations where we have two sides to the equality (e.g. $3 + 4 = \_\_\_ + 2$; $\_\_\_ + 1 = 4 – 3$, etc.). Understanding the concept of equality at the complex relational level is demonstrated in situations where the equals sign occurs repeatedly as a link between multiple expressions (e.g. $1 + 3 = \_\_\_ + 2 = \_\_\_\_ – 3 = \_\_\_$). The final sublevel of relational understanding requires the understanding of equality in the context of problem solving. This involves situations in which students are expected to solve specific problems using the balance method, i.e., jumping from one side of the equation to the other.

When it comes to the levels of understanding, it should be emphasised that Kieran (1981) believes that there is a certain misuse of the equals sign among students at all levels of learning, as well as that the operational interpretation of the equals sign begins in the preschool period. The same author argues that certain findings suggest that students’ initial understanding of the equals sign are based on their intuitive understanding of the equals sign as a “do something” symbol, or a symbol indicating where “the answer should go” even before they start formal education. Nevertheless, an intuitive concept formed in this way can be gradually transformed into the relational meaning of the equals sign, which is what teaching aims for, and which would later lay the foundations for learning more abstract content.

For this reason, the main idea behind this research is based on the need to investigate how students understand the equals sign and to examine potential differences in understanding between students of different age in order to identify potential difficulties in the development of this concept in junior primary school.

RESEARCH METHOD

The research goal is to identify the development of the concept of equality in line with the operationalised levels and recognise differences in understanding between students of different ages. Based on the research goal, the following research tasks were defined:

1) Determine the development of the equals sign among students at the operational level;
2) Determine the development of the equals sign among students at the relational level;
3) Determine the development of the equals sign among students at the relational level, in the context of real-world problem solving.

The research sample was selected among the population of students who attended the 2nd and 4th grade in two primary schools in Užice during 2021/2022 (Table 3). The sample was chosen by convenient sampling in order to obtain as objective results as possible. Five classes of second graders (N = 190) and five classes of fourth graders (N = 210) participated in the testing. The reason for choosing second grade students is the fact that the very first knowledge and experience of arithmetic and understanding of the equals sign are acquired in this period, and we wanted to see how firmly that knowledge foundation was built, and which level of understanding they reached. The fourth grade is the final grade in the first cycle of education, so there is a need for a comprehensive understanding of the equals sign as a symbol of mathematical equivalence. In addition, another reason for choosing fourth graders is the fact that similar research by McNeil (McNeil 2007) shows that operational understanding of the equals sign is still most firmly implanted among nine-year-olds.

Table 3. Sample of elementary school students

School	Second grade	Fourth grade	Total
School 1	116 48.33%	124 51.67%	240 100%
School 2	74 46.25%	86 53.75%	160 100%
Total	190 47.5%	210 52.5%	400 100%

The research was implemented using the testing technique. A knowledge test, which aims to determine the development of the equals sign among students, was created for this purpose. The test was created by incorporating models of math problems used by other researchers to illustrate the levels of development of the concept of equivalence (Knuth et al. 2008; Molina, Ambrose 2008; McAuliffe, Tambara, Simsek 2020; Rittle-Johnson, Alibali 1999; Rittle-Johnson, Matthews, Taylor, McEldoon 2011; Cockburn, Littler 2008).

The test was comprised of six problems. Examples of problems are listed in the results section. Students were tested for the duration of one school period and were only allowed to use a pencil to solve the problems. To examine the understanding of the equals sign at the *operational level,* we designed two problems, which aim to determine if students view the equals sign as a symbol of a general idea which translates to "calculate" or "find the solution". In the first problem, the students had the task to identify the sum that matches the sum 50 + 30, while in the second, they were asked to fill in the blank so that the left and the right side of the equals sign would be equivalent, whereby the expression was located only on one side of the equals sign.

The third and fourth problems involved equalities the solution of which required students to demonstrate that they possessed a developed relational understanding of the equals sign. In order to better examine the development of relational understanding of the equals sign, we distinguish two sublevels: basic and complex relational. The *basic relational level* involved equality-based problems in which operations were located on both sides of the equals sign. The *complex relational level* included equality-based problems with multiple equality signs. In this case, the equivalence involves a sequence of expressions with missing numbers. The fifth and the sixth problem referred to the understanding of the equals sign in the context of real-world problem solving, aiming to examine students' understanding of the equals sign in real-world problem solving.

Cronbach's alpha (0.802) indicates good reliability and internal consistency of the instrument used on this sample (Table 4).

Table 4. Cronbach alpha coefficient

Reliability Statistics		
Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
0.802	0.797	12

The tests were reviewed by two independent reviewers who have experience in this field, in order to achieve greater objectivity. The level of understanding of the equals sign was determined in relation to success in solving the given problems. The data obtained from conducting the test were processed quantitatively and qualitatively, and given in percentages in the tabular form. A chi-square test was used to test statistical significance of the differences between the variables. The obtained results were also analysed quantitatively, analysing typical errors and incorrect solutions.

RESULTS AND DISCUSSION

UNDERSTANDING THE EQUALS SIGN AT THE OPERATIONAL LEVEL

The first research task aimed to determine the development of the equals sign at the operational level. Looking at Table 5, we can see that both second graders (89.3%) and fourth graders (95.3%) were most successful in solving the problem that required them to calculate and enter the value of the expression: 80 + 20. Interestingly, they were less successful when asked to find the expression with the same value as the one provided (50 + 30). This indicates that students still largely view the equals sign as an instruction to calculate the result. The students were least successful when asked to find the value of the minuend and calculate the correct equality – second graders (66.3%) and fourth graders (79.7%). This shows that students do not view the expression as an independent entity/object, but only as an element to be calculated.

Table 5. Development of the equals sign at operational level

Task	Second grade		Fourth grade		Chi-square
	Successful	Unsuccessful	Successful	Unsuccessful	
Find the sum with the same value as the expression: 50 + 30. a) 50 + 80 b) 80 + 30 c) 40 + 40 d) None of the above	149 79.7%	38 20 3%	190 89.2%	23 10.8%	$\chi^2 = 6.987$, df = 1, p = 0.008
Insert the missing number. ______ = 80 + 20	167 89 3%	20 10.7%	203 95.3%	10 4.7%	$\chi^2 = 5.168$, df = 1, p = 0.018
40 = ______ – 20	124 66 3%	27 12.7%	186 87.3%	27 12.7%	$\chi^2 = 25.217$, df = 1, p = 0.000

If we compare the performance in relation to grade, we can conclude that fourth grade students were more successful than second grade students in every task. The value of the chi-square for each tested problem (Table 5) shows that there are differences in the performance between fourth grade and second grade students, and that they are statistically significant. Such results make perfect sense, especially considering the experience of the fourth graders with more abstract content, which helps them to transcend the operational level. The obtained results are consistent with similar research conducted in different countries (Jones et al. 2012; Knuth et al. 2006; Molina, Ambrose 2008; McAuliffe, Tambara, Simsek 2020; Fyfe et al. 2018; Capraro et al. 2010).

Some of the typical errors that students made when solving these problems are given in Figure 1.

Figure 1. Errors associated with operational level of understanding of the equals sign

In the first example shown in Figure 1, the student completely ignores the minus sign and focuses on the result by changing the sign to get the result that makes more sense to them. The second example of typical errors shown in Figure 1, illustrates the student's tendency to always expect the result on the right side of the equals sign. Similar research indicates that junior primary school students often find equalities, such as *100 = 80 + 20*, to be incorrect (Kieran 1981; Filloy, Rojano 1989; Carpenter, Levi 2000). Research by Knuth et al. (Knuth et al. 2008) reveals that most math textbooks present equalities as operations on the left side of the equals sign, while the right side is reserved for the results of the calculations, which may be one of the reasons why students think the way they do. Booth sees the solution to these problems in the fact that mathematics education requires various modifications of equalities so that the understanding of the equals sign would not be reduced to the expectation that the result of the expression is always located on the right side of it (Booth 1988).

UNDERSTANDING THE EQUALS SIGN AT THE RELATIONAL LEVEL

The second research task aimed to determine the development of the equals sign at the relational level, which comprises two sublevels – basic and complex. The obtained results show that the performance of the second graders in solving these tasks was under 50% (Table 6), and that they only demonstrated partial success in solving the following problem: 40 + 20 + 30 = 40 + ____, achieving 50.8%. Fourth graders were more successful in solving problems that examine the development of their basic relational understanding, except in one example. The problem: 12 + 23 = ____ + 26 turned out to be the biggest obstacle for both student groups, whereby only one in seven second graders managed to solve the problem correctly, and 40.4% of the respondents in the fourth grade. The reason for these results can be found in the operational understanding of the equals sign that is predominant among students. As a result, students put emphasis on the calculation, instead on the equivalence of the expressions on different sides of the equals sign.

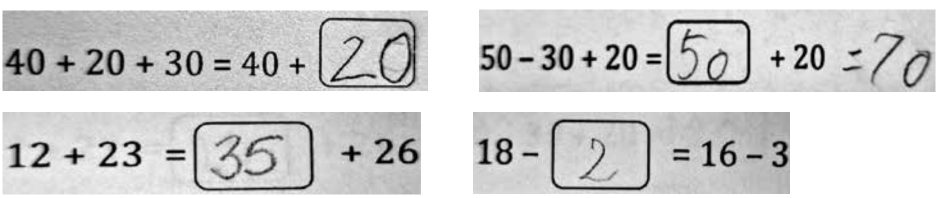
Table 6. Development of the equals sign at basic relational level

Task	*Basic relational level*				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
Insert the missing number. 40 + 20 + 30 = 40 + ____	95 50.8%	92 49.2%	149 70%	64 30%	$\chi^2 = 15.351$, df = 1, p = 0.000
50 – 30 + 20 = ____ + 20	64 34.2%	123 65 8%	112 52.6%	101 47.4%	$\chi^2 = 13.619$, df = 1, p = 0.000
12 + 23 = _____ + 26	24 12.8%	163 87 2%	86 40.4%	127 59.6%	$\chi^2 = 37.884$, df = 1, p = 0.000
18 – ____ = 16 – 3	51 27.3%	136 72.7%	109 51.2%	240 48.8%	$\chi^2 = 23.702$, df = 1, p = 0.000

If we compare the performance of second graders and fourth graders, we can see that there are statistically significant differences. The value of the chi-square test (Table 6) for each individual task shows that the differences are statistically significant, and that students of the fourth grade are more successful at the basic relational level of understanding of the equals sign.

We will highlight some typical errors that students made when solving these problems (Figure 2).

Figure 2. Errors associated with basic relational level of understanding of the equals sign



The first two examples (Figure 2) show that students ignore the value of the expressions on the left and right side of the equality, and focus on duplicating the expression, while the third and fourth example show operational understanding of the equals sign. It is obvious in these examples that students accept the equals sign as a command to “calculate” the result, thus ignoring the value of the expressions with unknown numbers, i.e., ignoring relational understanding of the equals sign. Similar results and typical errors in understanding of the equals sign have been obtained in other similar research around the world (Duncan 2015; McAuliffe, Tambara, Simsek 2020; Rittle-Johnson et al. 2011).

In addition to the basic relational level, we also wanted to determine students’ understanding of the equals sign in complex situations where the equals sign occurs multiple times. The analysis of the obtained results (Table 7) shows that only 4.8% of second grade students were successful in solving equations with multiple equals signs. Similarly, the percentage of fourth grade students who successfully solved this type of problem was also low (14.6% and 22.1%). Such results show insufficient development of relational understanding of the equals sign as a symbol of equivalence.

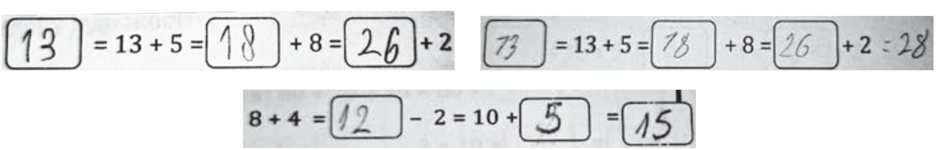
Table 7. Development of the equals sign at the complex relational level

Task	Complex relational level				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
Insert the missing number. 8 + 4 = ____ – 2 = 10 + ____ = _____	9 4.8%	178 95.2%	31 14.6%	182 85.4%	χ^2 = 10.499, df = 1, p = 0.001
_____ = 13 + 5 = ____ + 8 = ____+ 2	9 4.8%	178 95.2%	47 22.1%	166 77 9%	χ^2 = 24.618, df = 1, p = 0.000

Despite the fact that both second graders and fourth graders were unsuccessful in solving math problems of this type, if we compare the obtained results, we can see that fourth grade students achieve significantly better results compared to second graders, as confirmed by the values of the chi-square test (Table 7) for each individual problem.

Students made similar typical errors when solving equalities that feature multiple equals signs and require complex relational understanding, as when they solved problems that required basic relational level of understanding (Figure 3).

Figure 3. Errors associated with complex relational level of understanding of the equals sign



The research results show that both age groups of students were least successful in understanding the equals sign at the complex relational level (Table 7). The above examples indicate that students view the equals sign as an instruction "to calculate the result", i.e., that operational thinking dominates among students when they encounter the equals sign, that the manner in which they perform the operations is largely one-directional, and that they commonly fail to understand the equivalence between the expressions separated by the equals sign. This means that students have not developed relational understanding of the equals sign to a sufficient extent. Students are, thus, unable to highlight the interchangeability of the two sides of the equation (McNeil et al. 2006; Seo, Ginsburg 2003). In order to improve students' understanding of the equals sign, some researchers suggest to "take care with how you use the '=' sign when demonstrating complex problems with multiple steps. Use arrows if it is necessary to link the successive stages together" (Cockburn, Littler 2008:37).

The fact is that all students have demonstrated significant difficulties in relational understanding of the equals sign, but also that fourth grade students have a more developed relational understanding of the equals sign than second graders. Knuth et al. obtained similar results (Knuth, Stephens, McNeil, Alibali 2006). Their research shows that relational understanding of the equals sign (as a symbol of equivalence) improves over time, as well as that there is a link between the understanding of the equals sign, and the ability to solve equations in later stages of mathematics education. The same authors emphasise the fact that students who have had no prior experience with formal algebra are more successful in understanding and solving equations when older if they possess relational understanding of the equals sign.

UNDERSTANDING THE EQUALS SIGN AT THE RELATIONAL LEVEL IN THE CONTEXT OF REAL-WORLD PROBLEM SOLVING

The third research task referred to the students' performance in understanding the relational level of the equals sign in the context of real-world problem solving. Two problems were selected for this purpose:

1) Two bags of marbles are shown on the picture. The first bag holds 70, while the other holds 30 marbles. How many marbles should change places so that we have the same number of marbles in both bags?

2) There are 28 apples in one basket, and 24 in the other. Nena ate 2 apples from the second basket. How many apples should be transferred from the first to the second basket so as to have the same number of applies in both baskets?

Both problems came with an illustration of the problem situation, so that students would create a clearer picture of the given problem.

The obtained results show that 40.1% of second graders and 64.8% of the fourth graders successfully solved the marble problem (Table 8). When it comes to the apple problem, which is more complex, only a quarter of the second grade students (25.7%) achieved success. On the other hand, almost every other fourth grader (47.9%) successfully solved this type of problem. Compared to the results of the previous research task, we can see that both second graders and fourth graders are more successful in solving the real-world context marble problem in relation to the tasks that require relational understanding of the equals sign in a mathematical context. This fact must be taken into account, especially with regard to the need to improve the students' understanding of the equals sign, and in situations where it is possible to recognise positive aspects of different methodological approaches, such as real-world contexts.

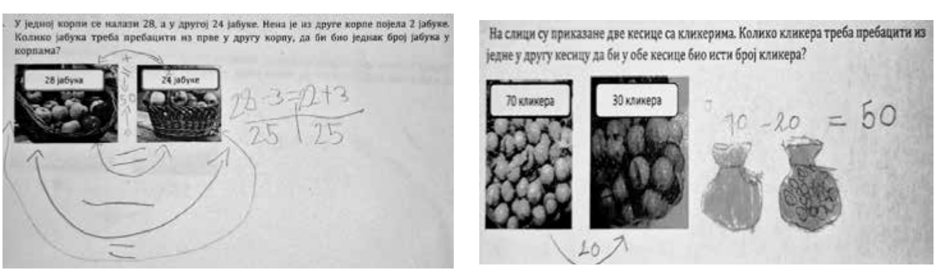
Table 8. Development of relational understanding of the equals sign in real-world problem solving

Tasks	*Relational understanding of the equals sign in real-world problem solving*				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
The first problem	75 40.1%	112 59.9%	138 64 8%	75 35 2%	χ^2 = 24.368, df = 1, p = 0.000
The second problem	48 25.7%	139 74 3%	102 47 9%	111 52.1%	χ^2 = 20.975, df = 1, p = 0.000

The value of the chi-test square in both problems shows that there are statistically significant differences between students of the second and the fourth grade. Fourth graders achieved significantly better results compared to the second graders in relational understanding of the equals sign.

Students used different strategies to solve the given tasks (Figure 4). Illustrative examples show the students' need to visualise problems and their use of drawings to facilitate the problem-solving process. Some researchers (Alexandrou-Leonidou, Philippou 2011) believe that the use of multiple visual representations reinforces one's understanding of the equals sign, highlighting the importance of visual and symbolic representations. In the process of solving problems of this type, students relied on different strategies and models that helped make the concept of equivalence more realistic and comprehensible. Visual representations can help in understanding the concept of equality, because they support structural concepts that make abstract ideas more tangible (Fagnant, Vlassis 2013).

Figure 4. Visual strategies for solving real-world context problems



The research by Milinković, Maričić and Đokić (2022) shows that students utilise different forms of visual and schematic representations in solving real-world context problems to present the equivalence of mathematical expressions. According to Dabić Boričić and Zeljić (Dabić Boričić, Zeljić 2021), the key factor of students' success in transforming equivalent expressions lies in developing the meaning of relationships through the process of modelling and other representations. Cockburn & Littler have a similar opinion (2008), arguing that it is necessary to "use concrete apparatus such as balances and visual images to represent a variety of number sentence structures with the 'unknown' on both the left and right-hand sides of the equals sign" (Cockburn, Littler 2008: 37).

CONCLUSION

Taking into account all of the obtained results, it could be argued that fourth grade students achieved significantly better results in the development of all levels of understanding of the equals sign. Despite the fact that the fourth graders were generally more successful, some particulars observable in the obtained results are worth mentioning:

- All students are more successful in operational than in relational understanding of the equals sign;
- Fourth grade students are significantly more successful in relational levels of understanding of the equals sign compared to second grade students;
- Despite being more successful than second grade students, fourth graders nonetheless demonstrate a significant percentage of failure at all levels of understanding;
- Almost one in every ten fourth grade students (except in one example) show that they have not even mastered operational level of understanding to the fullest extent;
- Improved understanding of the equals sign as a symbol of equivalence is evident in older students;
- Evident progress in understanding the equals sign as a symbol of equivalence in real-world context problems.

The research results show that students in junior primary school do not have sufficiently developed relational understanding of the equals sign. A large percentage of students, both in the second and the fourth grade, show that they perceive the equals sign in mathematical equalities as an operation, instead as a relationship that expresses the equivalence of the left and the right side of the equality. Regardless of the fact that there is progress, if the results across the tested classes are compared, the progress is still insufficient to help them understand the equals sign as a symbol of equivalence. The roots of this problem can be found in the fact that the syllabus and curriculum do not pay enough attention to the formation of this concept. There are no clearly defined guidelines or outcomes regarding the development of the concept of the equals sign in the *Rulebook on the Mathematics Syllabus for the First Cycle of Education in the Republic of Serbia* (2019), which only confirms that, despite its importance in elementary mathematics education, not nearly enough attention is paid to this content. Methodologists and practitioners should study the problems accompanying this concept in much more detail and prescribe guidelines that would lead to its proper development.

Considering the obtained results, one of the necessary requirements would certainly refer to introducing changes to the curricula and syllabi, as well as the textbooks, so as to underline the importance of studying this content. The key activity in this process would involve a revision and redesign of the examples and in-class activities, as well as learning examples that serve as the basis for building this concept.

Some of the research (Jacobs et al. 2007) shows that a large number of teachers are unaware of the differences between operational and relational understanding of the equals sign, which is why they tend to disregard the importance of building this concept. For this reason, more attention should be paid to the professional development of teachers through various forms of support, primarily to underline the problems in the correct development of the concept of equivalence and its importance in learning more complex math content. The latter is particularly important given the fact that relational understanding of the equals sign is crucial for the development of algebraic skills, including equation solving and algebraic thinking (Alibali et al. 2007; Jacobs et al. 2007; Kieran 1989; Knuth et al. 2006).

Students encounter various types of equalities from the very first grade, from the simplest arithmetic ones to equations with one unknown, equalities comprising expressions on both sides of the equals sign, etc. Different understandings of the equals sign must be developed simultaneously in teaching, and any operationalisation of the levels of understanding of the equals sign must not result in the interpretation of separate levels as discrete stages (McAuliffe, Tambara, Simsek 2020). In other words, separate levels of understanding are necessary and integral to the development of the concept of the equals sign, but in that process, adequate methods that will speed up the process must be chosen.

Paying insufficient attention to the construction of this concept may lead to undesirable understanding of the concept of equality, which lacks its fundamental property of equivalence. Therefore, it is essential to study children's understanding of this concept and the errors that occur in problem solving, which may result in the subsequent misunderstanding of more complex mathematical content. Some research suggests that students who understand the equals sign as an operational symbol achieve poorer results in algebra in later stages of education compared to those who nurture a relational understanding of the equals sign (Knuth et al. 2006).

Our research focused on equality-based problems of different levels of difficulty, most of which students encounter very seldom in math classes, which only makes the obtained results more valuable and objective. On the other hand, this research is limited due to the fact that the data were obtained through only one written test, so asking additional questions and conducting individual interviews with the students could shed more light on the students' understanding of this concept.

In conclusion, we would like to highlight some of the factors that affect the understanding of the equals sign, as proposed by Molina et al: (a) The cognitive demand of the operations involved in the sentence and, therefore, the students' mastery of arithmetic operations and their number sense; (b) Students' structure sense which includes the capacity to see an arithmetic or algebraic expression as a whole, to split an expression into sub-structures, to detect connections between the structures of different expressions and to recognize in an expression a known structure; (c) Students' knowledge of conventions of mathematic language (Molina, Castro, Castro 2009: 365). In addition to the factors listed above, there is one positive factor that stood out in this research: real-world context, as the basis of the relational understanding of the equals sign. In that sense, real-world context examples are the only ones that can be understood relationally, because the basis of the development of equivalence is found in the real and the tangible.

The research shows that elementary mathematics must focus on the development of relational understanding of the equals sign as one of its primary tasks, because it lays the groundwork for the successful mastering of more complex mathematical content.

REFERENCES

Alexandrou-Leonidou, Philippou (2011): V. Alexandrou-Leonidou, G. N. Philippou, Can they 'see' the equality?, In: M. Pytlak, T. Rowland, E. Swoboda, *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (*CERME 7)*, (410–419), Poland: University of Rzeszów.

Alibali, Knuth, Hattikudur, McNeil, Stephens (2007): M. W. Alibali, E. J. Knuth, S. Hattikudur, N. M. McNeil, A. C. Stephens, A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning*, 9, 221–247.

Baroody, Ginsburg (1983): A. J. Baroody, H. P. Ginsburg, The effects of instruction on children's understanding of the "equals" sign, *The Elementary School Journal*, 84(2), 199–212.

Behr, Erlwanger, Nichols (1976): M. Behr, S. Erlwanger, E. Nichols, *How Children View Equality Sentences*. Project for the Mathematical Development of Children. Technical Report No. 3, Florida State University. ERIC Document Reproduction Services No. ED144802

Behr, Erlwanger, Nichols (1980): M. Behr, S. Erlwanger, E. Nichols, How children view the equals sign, *Mathematics teaching*, 92(1), 13–15.

Booth (1988): L. R. Booth, Children's difficulties in beginning algebra, *The ideas of algebra*, K-12, 19, 20–32.

Capraro, Ding, Matteson, Capraro, Li (2007): M. M. Capraro, M. Ding, S. Matteson, R. M. Capraro, X. Li, Representational implications for understanding equivalence, *School Science and Mathematics*, 107(3), 86–88.

Carpenter, Levi (2000): T. P. Carpenter, L. Levi, *Developing conceptions of algebraic reasoning in the primary grades*, Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carpenter, Franke, Levi (2003): T. P. Carpenter, M. L. Franke, L. Levi, *Thinking mathematically: Integrating arithmetic and algebra in elementary school*, Portsmouth, NH: Heinemann.

Cobb (1987): P. Cobb, An investigation of young children's academic arithmetic contexts, *Educational Studies in Mathematics*, 18(2), 109–124.

Cockburn, Littler (2008): A. D. Cockburn, G. Littler, Mathematical Misconceptions, *A Guide for Primary Teacher*, SAGE.

Dabić Boričić, Zeljić (2021): M. Dabić Boričić, M. Zeljić, Modelovanje ekvivalencije matematičkih izraza u početnoj nastavi, *Inovacije u nastavi – časopis za savremenu nastavu*, 34(1), 30–43.

Duncan (2015): C. D. Duncan, *A study of mathematical equivalence: The importance of the equal sign* (Master's thesis, Southern University). LSU Digital Commons.

Ilić, Zeljić (2017): S. Ilić, M. Zeljić, Pravila stalnosti zbira i razlike kao osnova strategija računanja, *Inovacije u nastavi*, 30 (1), 55–66.

Jacobs, Franke, Carpenter, Levi, Battey (2007): V. R. Jacobs, M. L. Franke, T. P. Carpenter, L. Levi, D. Battey, Professional development focused on children's algebraic reasoning in elementary school, *Journal for Research in Mathematics Education*, 38(3), 258–288.

Jones, Inglis, Gilmore, Dowens (2012): I. Jones, M. Inglis, C. Gilmore, M. Dowens, Substitution and sameness: Two components of a relational conception of the equals sign, *Journal of Experimental Child Psychology*, 113(1), 166–176.

Kieran (1981): C. Kieran, Concepts associated with the equality symbol, *Educational Studies in Mathematics*, 12(3), 317–326.

Kieran (1989): C. Kieran, The early learning of algebra: A structural perspective, In: S. Wagner, C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (33–56), Reston, VA: NCTM

Knuth, Stephens, McNeil, Alibali (2006): E. Knuth, A. C. Stephens, N. M. McNeil, M. W. Alibali, Does understanding the equal sign matter? Evidence from solving equation, *Journal for Research in Mathematics Education*, 37(4), 297–312.

Knuth, Alibali, Hattikudar, McNeil, Stephens (2008): E. J. Knuth, M. W. Alibali, S. Hattikudar, N. M. McNeil, A. S. Stephens, The Importance of the Equal Sign Understanding in the Middle Grades, *Middle Teaching in the Middle School*, 13(9), 514–519.

Milinković, Maričić Đokić (2022): N. Milinković, S. Maričić, O. Đokić, The equals sign – the problem of early algebra learning and how to solve it, *Inovacije u nastavi*, XXXV(3), 26–41.

Molina, Ambrose (2008): M. Molina, R. Ambrose, From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking, *Focus on Learning Problems in Mathematics*, 30(1), 61–80.

Molina, Castro, Castro (2009): M. Molina, E. Castro, E. Castro, Elementary students' understanding of the equal sign in number sentences, *Education & Psychology*, 17, 341–368.

McAuliffe, Tambara, Simsek (2020): S. McAuliffe, C. Tambara, E. Simsek, Young students' understanding of mathematical equivalence across different schools in South Africa, *South African Journal of Childhood Education*, 10(1), 1–8.

McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur, et al. (2006): N. M. McNeil, L. Grandau, E. J. Knuth, M. W. Alibali, A. S. Stephens, S. Hattikudur, et al., Middle-school students' understanding of the equal sign: The books they read can't help, *Cognition & Instruction*, 24(3), 367–385.

McNeil (2007): N. M. McNeil, U-shaped development in math: 7-year-olds outperform 9-year-olds on equivalence problems, *Developmental Psychology*, 43, 687–695.

McNeil (2008): N. M: McNeil, Limitations to teaching children 2 + 2 = 4: Typical arithmetic problems can hinder learning of mathematical equivalence, *Child Development*, 79(5), 1524–1537.

McNeil, Weinberg, Hattikudur, Stephens, Asquith, Knuth, Alibali (2010): N. M. McNeil, A. Weinberg, S. Hattikudur, A. C. Stephens, P. Asquith, E. J. Knuth, M. W. Alibali, A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions, *Journal of Educational Psychology*, 102(3), 625–634.

Parslow-Williams, Cockburn (2008): P. Parslow-Williams, A. D. Cockburn, Equality: getting the right balance, In: A. D. Cockburn, G. Littler (eds.), *Mathematical Misconceptions* (23–38), SAGE.

Pravilnik (2019). Правилник о плану и програму наставе и учења за четврти разред основног образовања и васпитања, *Просветни гласник, Службени гласник Републике Србије*, бр. 11/2019.

Rittle-Johnson, Alibali (1999): B. Rittle-Johnson, M. W. Alibali, Conceptual and procedural knowledge of mathematics: Does one lead to the other?, *Journal of Educational Psychology*, 91, 175–189.

Rittle-Johnson, Matthews, Taylor, McEldoon (2011): B. Rittle-Johnson, P. G. Matthews, R. S. Taylor, K. L. McEldoon, Assessing knowledge of mathematical equivalence: A construct-modeling approach, *Journal of Educational Psychology*, 103(1), 85.

Seo, Ginsburg (2003): K. H. Seo, H. P. Ginsburg, "You've got to carefully read the math sentence…": Classroom context and children's interpretations of the equals sign, In: A. J. Baroody, A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (161–187), Lawrence Erlbaum Associates, Mahwah, NJ.

Fagnant, Vlassis (2013): A. Fagnant, J. Vlassis, Schematic representations in arithmetical problem solving: Analysis of their impact on grade 4 students, *Educational Studies in Mathematics*, 84(1), 149–168.

Falkner, Levi, Carpenter (1999): K. Falkner, L. Levi, T. Carpenter, Early Childhood Corner: Children's Understanding of Equality: A Foundation for Algebra, *Teaching children mathematics*, 6(4), 232–236.

Filloy, Rojano (1989): E. Filloy, T. Rojano, Solving equations: The transition from arithmetic to algebra, *For the learning of mathematics*, 9(2), 19–25.

Fyfe, Matthews, Amsel, McEldoon, McNeil (2018): E. R. Fyfe, P. G. Matthews, E. Amsel, K. L. McEldoon, N. M. McNeil, Assessing formal knowledge of math equivalence among algebra and pre-algebra students, *Journal of Educational Psychology*, 110(1), 87–101.

Ненад С. Милинковић
Сања М. Маричић
Универзитет у Крагујевцу, Педагошки факултет у Ужицу
Бојан Д. Лазић
Универзитет у Новом Саду, Педагошки факултет у Сомбору

ПРОБЛЕМ (НЕ)РАЗУМЕВАЊА ЗНАКА ЈЕДНАКОСТИ У МЛАЂИМ РАЗРЕДИМА ОСНОВНЕ ШКОЛЕ

Резиме: Правилно разумевање знака једнакости представља кључну основу за разумевање аритметике, али и основни појам важан за учење других области математике. Истраживања широм света наводе проблеме правилног разумевања знака једнакости, при чему се у први план истиче ограничавајући поглед на знак једнакости као наредбу „израчунај". Циљ истраживања био је да се идентификује развијеност појма једнакости према операционализованим нивоима (операциони, релациони и релациони у контексту решавања реалног проблема) и утврде разлике у разумевању између ученика другог (N = 190) и четвртог разреда (N = 210) основне школе. Истраживање је реализовано техником тестирања. Резултати истраживања показују да ученици немају довољно развијено релационо разумевање знака једнакости, већ да доминира његово операционо схватање. Боље разумевање знака једнакости на сваком нивоу разумевања показали су ученици четвртог разреда.

Кључне речи: знак једнакости, еквивалентност, операционо разумевање, релационо разумевање, математика, математичко образовање.

Marijana M. Gorjanac Ranitović
University of Novi Sad, Faculty of Education in Sombor
Department of Mathematics and Mathematics Teaching Methodology

Mirjana T. Maričić
Department of Natural Sciences and Management in Education

Mia R. Marić
Department of Social Sciences

Aleksandar V. Petojević

Snežana S. Gordić
Department of Mathematics and Mathematics Teaching Methodology

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TEACHERS' PERCEPTIONS OF INSTRUCTIONAL GUIDANCE IN ONLINE MATHEMATICS TEACHING[1]

Abstract: The purpose of the research was to examine teachers' perceptions of the requirements and benefits of using indirect versus direct instruction in online mathematics teaching and its relation with socio-educational variables. Also, it is examined whether, compared to other subjects, teachers more often apply a certain type of instruction in mathematics classes, and what teaching materials and tools for communication they use when applying direct and indirect instruction in online mathematics teaching. The results showed that teachers perceive the benefits and requirements of indirect instruction compared to direct instruction, and this perception is a slightly determined by levels of their education and work experience. About half of teachers, use direct instruction more often in online mathematics classes, compared to the other subjects. They use a wide range of teaching materials and tools for communication. The results have implications for the further professional development of teachers in the domain of using direct and indirect instructions in mathematics teaching.

Keywords: direct and indirect instruction, online mathematics teaching, primary education, teachers' perceptions.

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ranitovicm@uns.ac.rs

INTRODUCTION

The influence of instructional guidance in the teaching process is still not fully clarified in the related literature. In the scientific community, there are no agreed upon positions regarding how much instructional guidance should be provided in the learning process i.e., when it is necessary to provide explicit/direct support, and when only to guide independent student activities (Lee, Anderson 2013). Moreover, the results of the research are often completely opposite. The problem of determining the appropriate and optimal instructional guidance in mathematics teaching is a very complex problem that needs to be viewed from different perspectives. With this work, we want to make a contribution to the current research that is being carried out in order to determine the necessary level of instructional guidance in mathematics classes, while taking into account all the complex factors that affect the learning process. Our current focus is on online mathematics instruction, due to the expectation that online instruction will continue to have a significant place in the educational system.

THEORETICAL BASIS OF RESEARCH

When talking about instructional guidance, direct and indirect instructions are most often mentioned i.e., direct and indirect instructional guidance. To avoid terminological confusion, we will first consider the concept of direct instruction. In the scientific literature, the teaching model referred to as *DI* ("capital *DI*") and the method of instructive guidance referred to as *di* ("little *di*") are denoted by the same term (Nifdi 2022; Stockard et al. 2018). *DI* is an educational program (instructional model) that was developed in the 1960s by Zig Engelmann and his colleagues based on the assumption that for effective learning it is necessary to provide precise instructions, use well-chosen, sequenced examples, and that the transition to new concepts is possible only when the previous key concepts are mastered (Stockard et al. 2018). The term direct instruction (*di*) was introduced in 1976 by Rosenshine to define teacher strategies that are significantly related to student achievement (Nifdi 2022). Today, direct instruction refers to educational programs that apply explicit (direct) instruction (Stockard et al. 2018: 480), and also instructional guidance with full explanations of concepts, procedures, and problem-solving strategies (Kirschner, Sweller, Clark 2006). In this paper, by direct instruction (further *di*), we mean a highly guided instructional approach organized around key concepts within a certain teaching content that the teacher presents step by step, providing students with all the necessary explanations, ready-made answers, independent practice with explicit feedback, and check-ups on what has been learned (Cvjetićanin, Maričić 2022). Direct instructional guidance implies the decisive role of the teacher in preparing and providing all the necessary information, presenting models, facts, rules and procedures in the most explicit way (Aung, Khine 2020). From the founding of indirect instruction, which took place in the mid-1960s when it was scientifically proven that unguided learning does not produce the desired results, to its modern understanding as an approach that focuses on students and as much as possible engages their independence, productivity, imagination, creativity, etc., indirect instruction is understood as a different approach for different researchers (Loibi, Rummel 2013; Kittell 1957). This difference is reflected precisely in the optimal dosage of the offered guidelines, i.e., the quantitative determination of the minimum amount of guidance and its appropriate implementation in the teaching process (Maričić et al. 2022a; Maričić et al. 2022b; Matlen, Klahr 2013). Indirect instruction (further *ii*) means a less guided instructional approach organized around key concepts within a certain teaching content, which are presented to students step by step in the form of tasks or problems that they should realize or solve independently. Students should find the necessary explanations, then systematize, explain, and present what they have learned (Cvjetićanin, Maričić 2022; Eysnik, De Jong 2012). During this process, students are offered guidance in the form of instructions that can be embedded in the presented tasks, in the form of references to additional sources of knowledge, in the form of implicit questions, or in the form of hints (Dignath, Veenman 2021).

We view instructional guidance as a continuum, at one end of which there is direct instructional guidance in which the teacher plays a dominant role, and at the other end there is minimal instructional guidance that enables students to independently and freely explore and construct knowledge. Between these two extremes there is room for finding a balance in the application of direct and indirect guidance of students in the process of acquiring knowledge. The debate about the advantages of one or another model of instructional guidance has been going on for more than 50 years. The arguments and evidence presented in these discussions indicate the complexity of the process of instructional guidance and the need to look at the problem from different points of view (Aung et al. 2020; Upu, Buhari 2014; Lee, Anderson 2013; Kirschner et al. 2006; Mayer 2004). In the category of minimally guided instruction, Kirschner includes problem-based learning and inquiry, and experiential and constructivist learning, without making an essential difference between these teaching approaches. According to Kirschner, minimally guided instruction cannot be effective primarily because it does not respect the human cognitive architecture and “learners should be explicitly shown what to do and how to do it” when dealing with novel information (Kirschner et al. 2006). Among the disadvantages of Kirchner’s observation of teaching guidance, the following stand out: neglecting the role of motivation and the fact that it is very important that the studied contents make sense for the students themselves, identifying different teaching models with a minimally guided approach, and favoring instructional guidance that develops lower cognitive levels (Hmelo-Silver, Duncan, Chinn 2007; Kuhn 2007; Schmidt et al. 2007). The teaching models that Kirchner identifies with minimally guided instruction have proven their effectiveness with indirect instructional guidance and are based on the assumption that knowledge is built on the basis of personal experience (Kalyuga et al. 2001; Dean, Kuhn 2007; Alferi et al. 2011; Maričić et al. 2022a; Cvjetićanin et al. 2022; Brunstein, Betts, Anderson 2009). Therefore, students are not left alone in acquiring knowledge but receive support in the form of indirect instructions, peer interaction and assistance, and the use of technology (Upu et al. 2014; Hmelo-Silver et al. 2007).

The goal of learning mathematics is not only the mastery of mathematical concepts and the development of abstract, logical, and critical thinking, but also the development of "skills of knowledge acquisition – skills that equip a new generation to learn what they need to know to adapt flexibly to continually changing […]" (Kuhn 2007). Recent research points to the need for balanced instructional guidance in teaching mathematics (Aung et al. 2020; Upu et al. 2014; Oladayo, Oladayo 2012; DeCaro, Rittle-Johnson 2012; Jones, Southern 2003). Today, the prevailing understanding is that in teaching mathematics it is necessary to apply both direct and indirect instructional guidance, and that the greater challenge is to achieve a balance and the right sequence between them.

The COVID-19 pandemic instigated a series of innovative approaches to teaching and learning, including so-called online teaching, which is based on the use of modern educational technologies. The effectiveness of planning, preparation, and implementation of online mathematics lessons also depends on the teachers' perceptions of instructional guidance in online mathematics teaching. Online teaching means a form of distance education in an online environment and refers to situations in which the presence of the Internet supports the learning process (Fakhrunisa, Prabawanto 2020; Appana 2008). This learning does not depend on the physical or virtual location of the teacher and the student, and the teaching content is delivered online.

Regarding the aspect of instructive guidance, online mathematics teaching, as well as regular teaching, can be realized in two basic ways: with the application of direct and indirect instruction. It was confirmed that teachers have positive perceptions about the application of indirect instruction in learning mathematical content with the application of modern technology (Warner, Kaur 2017). The teachers stated that although the teaching of mathematical contents with direct instruction is easier, the results from the teaching with indirect instruction are much more pleasant. The teachers also encountered certain difficulties in their work, which are related to the technical side of working in the computer room, as well as to the fact that it is necessary to put the students in a position to think and thus adopt mathematical concepts, instead of just giving them a lot of examples, in order to prepare them for the test (Warner et al. 2017; Trybus 2013). Teachers' perceptions of student engagement in online teaching of geometric mathematics content (animated geometry) were examined (Aaron, Herlost 2015). It was found that teachers pay the most attention to sources that students can use correctly or incorrectly in their work, while they pay little attention to operations that students could apply in their work, as well as the goals that students should achieve while solving tasks. Inequality in the learning of mathematics content is also present in regular classes, but research has shown that it is significantly increased in online mathematics classes (Yılmaz et al. 2021). The results of this research are consistent with the results of numerous studies on this topic (Baysu, Ağırdağ 2019; Hohlfeld et al. 2017; Özdemir 2016). In addition, results confirmed that student engagement and interaction are not of the same quality during regular and online teaching (Yılmaz et al. 2021). The teachers declared that they encountered difficulties in applying various strategies and mechanisms for providing support and guidance to students, which affected their engagement and mathematical thinking. These data point to the fact that indirect guidance during online mathematics teaching has proven to be quite unsuccessful. Regarding the importance of using digital tools in promoting students' cognitive development, the teachers who are prepared for online teaching declared that they attach more importance to the research of mathematical concepts, to the technical demonstration, as well as to the discussion about what is shown on the screen, while the teachers who are not prepared for online teaching attach greater importance to the visualization of mathematical concepts and their mutual connection, to its explanation and to explaining mathematical representations (Guerrero-Ortiz, Huincahue 2020). The application that teachers used at the beginning of online classes was *WhatsApp*. Teachers mostly used videos, digital documents, and tutorials from the Internet. Teachers stated that they sent learning material to students in the form of modules, videos, and other materials, after which they directed students to online discussions or gave them online quizzes. Of the applications teachers most often used, *WhatsApp* and *Google Classroom* were used most for transferring materials; *Zoom*, *Google Meeting* and *Jitsi* were used for holding discussions; *Google Forms* and online quizzes helped to check what students; the most used application was *WhatsApp* (Guerrero-Ortiz et al. 2020; Fakhrunisa et al. 2020). For the disadvantages of online teaching, the teachers pointed out the following: teachers' readiness to launch applications and students' difficulty in using them, ignorance of the possibilities for more effective online tools that students can use, limitations in achieving learning that requires mathematical thinking, limitations in providing and receiving feedback, inability of some students to control the freedom with their time, and the need for direct guidance (for weaker students). For the advantages of online teaching, teachers pointed out: encouraging students to work independently, encouraging students and teachers to master the use of modern technologies, more flexible study time, adaptation of students to a more creative approach in performing tasks, and better storage of material that remains after the lesson. From the above facts, it is concluded that in the initial stages of online teaching of mathematics, it is necessary to offer professional training to teachers for working with certain digital tools, as well as direct instruction to students so that they can use all the benefits of this learning mode and acquire the necessary knowledge in this way. This would also improve communication between teachers and students. After a certain amount of time and the acquisition of adequate skills for working with digital tools, direct instruction could increasingly give way to indirect instruction, which would contribute to student independence in work and a more creative approach in performing their duties.

METHODOLOGY

THE PROBLEM, OBJECTIVES, AND METHODS

The subject of this research is the perception of the specific characteristics of the application of indirect versus direct instruction in online mathematics classes from the teachers' perspective. The research problem can be formulated in the form of the following question: how do teachers perceive the use of instructional guidance in online mathematics classes?

The main goal of the research is to examine the teachers' perception of the requirements and benefits of using indirect versus direct instruction in online teaching of mathematics. In addition, one of the objectives was to examine the impact of socio-educational variables, specifically teachers' work environment, level of education, and years of work experience on the way teachers perceive the application of indirect versus direct instruction in online mathematics classes. It was also determined whether, compared to other subjects, teachers more often apply a certain type of instruction in mathematics classes, as well as what teaching materials and tools for communication they use when applying direct and indirect instruction in online mathematics classes.

Theoretical analysis was used for explanation of the key concepts. The following methods were used in the research: descriptive-analytical method and methods of inferential statistics.

SAMPLE

The sample used in this research consists of 228 teachers in the first cycle of primary education in the Republic of Serbia. An overview of the characteristics of the sample can be found in Table 1. Currently, 2 respondents are pursuing professional studies, 9 respondents are pursuing academic studies, 10 are in master's programs and 11 are in doctoral programs.

Table 1. An overview of the characteristics of the sample

Gender		Environment		Level of education		Years of working experience	
Male	18	Urban	120	Professional studies	26	Less or equal to 10	43
Female	206	Rural	101	Academic studies	121	From 11 to 20	30
Non-binary	1	No answer	7	Specialist studies	3	From 21 to 30	101
No answer	3			Master studies	74	More or equal to 31	52
				Ph.D studies	1	No answer	2
				No answer	3		

INSTRUMENT AND PROCEDURE

For the purpose of our research, a questionnaire *Direct and indirect instruction in classroom mathematics* was created, which contains 13 questions as a part of our wider research. The first part of the questionnaire included socio-educational characteristics of the chosen sample, such as the environment in which the teachers work, their level of education, and their work experience. In the second part of the questionnaire, the respondents could express their agreement with the statements on a five-point Likert scale. For the purposes of data processing, respondents' answers were assigned values from 1 ("do not agree at all") to 5 ("totally agree"). In the closed-ended questions, the respondents could choose which type of instruction they use most often in online mathematics classes compared to other subjects. The questionnaire was created in an online format and distributed by sending a link through which respondents filled in the questionnaire electronically. The questionnaire was sent to all elementary schools in the Republic of Serbia, with the indication that is intended for teachers in first cycle of primary education. The data was processed in the statistical package IBM SPSS for Windows, version 20.

RESULTS

The first task of the research was to examine teachers' perceptions of the application of indirect instruction (*ii*), in relation to direct instruction (*di*), in online mathematics teaching. The first group of items refers to the requirements for the application of *ii* in relation to *di* in online mathematics teaching: The application of *ii* in relation to *di* in online mathematics teaching requires:

1. a more active role and greater engagement of the teacher (Item code R1),
2. greater methodological competence of the teacher (R2),
3. more time for teacher preparation (R3),
4. more material and technical resources (R4),
5. is more complex and can represent a professional challenge for teachers (R5),
6. a more active role and greater engagement of students (R6).

Descriptive indicators of the teacher's perception of the requirements for application of *ii* in relation to *di* in online mathematics teaching are presented in Table 2. From Table 2 we can see that the teachers least agree with the statement that the application of *ii* in relation to *di* in online mathematics teaching requires a more active role and greater engagement of students; the teachers most agree with the statements that the application of *ii* in relation to *di* in online mathematics teaching requires more material and technical resources and that the application of *ii* compared to *di* in online mathematics teaching requires more time for teacher preparation.

Table 2. Descriptive indicators of the teacher's perception of the requirements for application of *ii* in relation to *di* in online mathematics teaching

Items	*N*	1	2	3	4	5	*M*	*SD**
R1	222	3%	7.7%	20.7%	28.4%	40.1%	3.95	1.096
R2	222	1.4%	1.5%	18.9%	29.7%	40.5%	3.99	1.049
R3	224	2.2%	8.0%	16.5%	26.3%	47%	4.08	1.075
R4	224	1.8%	5.8%	16.5%	26.3%	47%	4.09	1.013
R5	219	1.4%	7.8%	18.3%	31.9%	40.6%	4.03	1.013
R6	225	6.2%	12.4%	19.6%	30.2%	31.6%	3.68	1.215

*Standard deviation

The second group of items refers to the benefits of indirect instruction compared to direct instruction in online mathematics teaching. The application of indirect instruction in relation to direct instruction in online mathematics teaching…

1. better equips students for independent work (Item code C1),
2. is more effective in terms of developing student competencies (C2),
3. contributes to the quality of interaction with students (C3),
4. encourages students' interest in teaching (C4).

Descriptive indicators of the teacher's perception of the contribution for application of indirect instruction in relation to direct instruction in online mathematics teaching are presented in Table 3. From Table 3 we conclude that the teachers least agree with the statement that the application of indirect instruction in relation to direct instruction in online mathematics teaching contributes to the quality of interaction with student; the teachers most agree with the statement that the application of indirect instruction in relation to direct instruction in online mathematics teaching better equips students for independent work.

Table 3. Descriptive indicators of the teacher's perception of contribution for application of *ii* in relation to *di* in online mathematics teaching

Items	N	1	2	3	4	5	M	SD
C1	225	6.2%	12.4%	19.6%	30.2%	31.6%	3.68	1.215
C2	223	6.3%	13.5%	20.2%	32.7%	27.3%	3.61	1.198
C3	222	5.9%	18.0%	23.4%	28%	24.7%	3.48	1.210
C4	225	7.2%	13.8%	24.0%	27.5%	27.5%	3.55	1.228

The Kolmogorov–Smirnov test (Sig. = .000) and Shapiro–Wilk test (Sig. = .000), as well as the shape of the histogram, showed that the scores on the all items were not normally distributed. Therefore, we used non-parametric methods for data analysis. The Mann–Whitney U test showed that there is no statistically significant difference on the items R1–R6 (p = .123–.779) and C1–C4 (p =.356–.919) in relation to the environment where the teacher works. The Jonckheere–Terpstra test for ordered alternatives revealed statistically significant differences on items R4 (T_{JT} = 8225000, z = 2.089, p = .037, r = 0.14 small effect) and C1 (T_{JT} = 8287500, z = 1.987, p = .047, r = 0.13 small effect) in relation to level of education. Groups of professional studies, academic studies, and specialist studies have a median of 4; groups master studies and Ph.D studies have a median of 5. The influence of the level of education on items R1–R3, R5, R6 (p = .128–.689) and C2–C4 (p = .119–.575) is not statistically significant. The Jonckheere–Terpstra test for ordered alternatives revealed statistically significant differences on item R6 (T_{JT} = 8447000, z = -1.982, p = .047, r = 0.13 small effect) in relation to years of working experience (Gp1 1–10 years, n = 42, Md = 4.28; Gp2 11–18 years, n = 26, Md = 4.26; Gp3 19–25 years, n = 32, Md = 4.09; Gp4 26–33 years, n = 82, Md = 4.02; Gr5 34+ years, n = 41, Md = 4.07).

In relation to other subjects in online mathematics teaching, 52.2% of teachers more often apply direct instruction, 14.7% of teachers more often apply indirect instruction, and 33% of teachers state that there is no difference compared to other subjects.

Furthermore, the teachers had to rate the extent to which they used the offered tools for communication with students and the implementation of online mathematics teaching using direct instruction on a scale from 1 ("did not use it") to 5 ("used it to a great extent"). The descriptive indicators are shown in Table 4. The most frequently used tool for communication with students in online mathematics teaching with direct and indirect instructions is Viber. Teachers reported that they have also used the RTS platform (sample lessons recorded on TV), *ClassDojo*, *MIT AppInventor*, *e-classroom*, *Messenger*, *Google Meet*, and *Discord* for both direct and indirect instruction.

Table 4. Descriptive indicators of the use of tools for communication with students and the implementation of online mathematics teaching when applying *di* and *ii*

Tool	Instr.	*N*	1	2	3	4	5	*M*	*SD*
Google	di	221	21.7%	9%	12.7%	15.8%	40.8%	3.45	1.599
Classroom	ii	217	27.2%	10.6%	10.6%	12.4%	39.2%	3.26	1.683
My Tesla	di	200	76.5%	7.5%	8.5%	3%	4.5%	1.52	1.070
classroom EDU	ii	200	77%	6%	6%	6.5%	4.5%	1.56	1.142
Microsoft Teams	di	197	3.4%	62.7%	12.3%	11.3%	7.4%	1.71	1.126
	ii	197	67%	11.2%	9.6%	7.1%	5.1%	1.72	1.199
Ed-modo	di	198	74.7%	10.1%	6.1%	3.5%	5.6%	1.55	1.120
	ii	200	77%	8.5%	6%	3.5%	5%	1.51	1.089
Jitsi	di	195	87.2%	5.6%	5.1%	2.1%	0%	1.22	0.632
	ii	197	85.8%	6.1%	5.1%	2.5%	0.5%	1.26	0.714
Online quizzes	di	211	30.3%	14.7%	22.7%	15.6%	16.6%	2.73	1.456
	ii	211	33.6%	16.1%	17.1%	14.2%	19%	2.69	1.523
Moodle	di	196	73.5%	10.7%	6.1%	6.6%	3.1%	1.55	1.068
	ii	199	76.4%	9%	5.5%	4.5%	4.5%	1.52	1.086
Google Drive	di	197	49.7%	11.7%	16.8%	11.2%	10.7%	2.21	1.427
	ii	200	59%	12.5%	11.5%	8.5%	8.5%	1.95	1.348
e-mail	di	211	19.9%	13.7%	12.3%	19%	35.1%	3.36	1.553
	ii	205	25.4%	13.2%	15.1%	22.4%	23.9%	3.06	1.528
Zoom	di	196	46.4%	11.2%	15.3%	12.8%	14.3%	2.37	1.512
	ii	203	54.2%	10.3%	12.3%	10.3%	12.8%	2.17	1.488
Skype	di	192	62.5%	13%	7.3%	7.8%	9.4%	1.89	1.360
	ii	197	67%	8.6%	6.6%	7.1%	10.7%	1.86	1.403
Viber	di	222	6.3%	7.2%	9.9%	13.5%	63.1%	4.20	1.246
	ii	217	6%	7.4%	17.5%	18%	51.2%	4.01	1.236
Social networks	di	199	54.8%	9.5%	12.6%	11.6%	11.6%	2.16	1.471
	ii	202	56.4%	10.4%	11.9%	8.4%	12.9%	2.11	1.476
Talking on the	di	214	17.8%	10.7%	21.5%	15.4%	34.6%	3.38	1.490
phone	ii	212	19.3%	10.8%	22.6%	14.6%	32.5%	3.30	1.500

Teachers were directed to choose which type of teaching materials they offer their students when applying direct instruction within online mathematics classes. PowerPoint, Prezi, and other types of presentations were chosen by 72.8% of teachers; text materials were chosen by 83.3% of teachers; additional content and explanations along with text materials were chosen by 75.4% of teachers; text materials for practice were chosen by 75.4% of teachers; video materials were chosen by 71.9% of teachers; audio materials are used by 33.3% of teachers; simulations were chosen by 18.4% of teachers; links to useful content or websites were chosen by 61.8% of teachers; and charts, diagrams, illustrations and similar tools were chosen by 74.1% of teachers.

Also, teachers were directed to choose which type of teaching materials they offer their students when applying indirect instruction within online mathematics classes. PowerPoint, Prezi, and other types of presentations were chosen by 66.2% of teachers; text materials were chosen by 69.7% of teachers; additional content and explanations along with text materials were chosen by 58.8% of teachers; text materials for practice were chosen by 67.1% of teachers; video materials were chosen by 68.8% of teachers; audio materials were chosen by 32.4% of teachers; simulations were chosen by 17.1% of teachers; links to useful content or websites were chosen by 65.8% of teachers; and charts, diagrams, illustrations and similar tools were chosen by 67.5% of teachers. Teachers stated that they still use prepared games and recorded lessons.

DISCUSSION

Given that the method and mode of instruction represent key elements in the effectiveness of the realization of teaching goals, as well as the current importance of online teaching during the COVID-19 pandemic, and the increasingly frequent use of modern technologies in teaching (Fakhrunisa et al. 2020; Kopas-Vukašinović, Mihajlović, Miljković 2021; Singh, Thurman 2019), it was important to examine how teachers perceive the application of direct and indirect instruction in online mathematics classes and with which socio-educational factors their answers are related. Also, our research was particularly focused on determining whether they choose a certain type of instruction more often in the mathematics class and to examine which teaching aids and communication tools they predominantly use for each type of instruction.

When it comes to the perception of the benefits of indirect instruction compared to direct instruction, this research showed that teachers significantly perceive more positive aspects of indirect instruction compared to direct instruction, which corresponds to previous studies that dealt with the issue of indirect instruction in mathematics teaching (Aaron et al. 2015; Warner et al. 2017). In this research, it was shown that as the greatest advantage of the application of indirect instruction compared to direct instruction is that teachers see the preparation of students for independent work and a positive impact on the development of student competencies. Also, to a lesser extent, they perceive that indirect instruction contributes to encouraging students' interests in the content, as well as to a better quality of interaction with students. This is especially important because teachers often state that communication is a problem when implementing online classes (Hohlfeld et al. 2017; Yılmaz et al. 2021), and the application of indirect instruction in this sense can be singled out as one of the potential ways to partially overcome this problem.

In the context of the requirements of applying indirect instruction compared to direct instruction in online mathematics classes, teachers mostly assess that this type of instruction requires greater material and technical resources, more time for lesson preparation, and that its application is generally more complex and represents a greater a challenge for teachers. This is consistent with previous studies that have dealt with the potential limitations and disadvantages of using indirect versus direct instruction in online teaching (Trybus 2013; Warner et al. 2017), which indicated that the realization of mathematical content with direct instruction is simpler, and that the preparation requires much less time. Previous studies also determined that with indirect instruction it is more difficult to get students to think independently, and that this type of work in an online environment requires additional material and technical support, which proved to be particularly problematic when working with students who come from socio-economically disadvantaged backgrounds (Baysu et al. 2019; Özdemir 2016). In addition to the above, teachers – to a significant extent – perceive that the application of indirect instruction in online mathematics classes requires greater methodological competence and a more active role for teachers and students. This also corresponds to the findings of previous studies, which indicate that this type of work requires greater methodological and technical competence (Fakhrunisa et al. 2020; Guerrero-Ortiz et al. 2020). The findings of this research indicate that it is necessary to work on the continuous development of the methodological and technological competences of teachers and to provide them with appropriate material and technical support, in order to apply indirect instructions as efficiently as possible and enable the most active role of students in online mathematics classes.

This research has shown that teachers from urban and rural areas equally perceive the benefits and requirements of applying direct and indirect instruction in online mathematics teaching. It has been noticed that students of a higher educational level (with completed master's and doctoral studies) perceive slightly more intensively that the application of indirect instruction better equips students for independent work, but also that it requires greater material and technical resources. It is possible that they are better informed about the characteristics of the application of indirect instruction in teaching, thanks to the additional education they have acquired, although it should be taken into account that these are small perceptual differences. Also, it was shown that teachers with less work experience perceive to a slightly greater extent that the application of indirect instruction requires a more active role and intensive engagement of students in online mathematics teaching. This may be a consequence of the fact that they completed their studies more recently, in accordance with contemporary educational paradigms, which have increasingly focused on the active role of the student, in contrast to traditional teaching where the student is more passive. Further, less experienced teachers tended to have higher expectations for students' performance, but it should be highlighted that this tendency was slight. It could be expected to a certain extent that the acquired education and work experience would shape the teachers' perceptions when it comes to the application of instructions in online teaching, so those results are consistent with previous findings (Trybus 2013; Warner et al. 2017).

The results of this research showed that approximately half of teachers, compared to other subjects, use direct instruction more often in online mathematic teaching while about a third of teachers equally apply direct and indirect instruction across subjects. Only about 14% of teachers indicated that they use indirect instruction more often than direct instruction in online mathematics classes. It can be assumed that the reason for this result is that teachers perceive the implementation of indirect instructions in online mathematics classes as more complex, as requiring more time, and as requiring additional material and technical resources, which they often do not have (Akram et al. 2022). This suggests that it is necessary to work on strengthening the competences of teachers and to improve the teaching-technological infrastructure in schools.

Regarding the use of tools for communication in online mathematics classes, teachers reported that they predominantly used *Viber*, *Google Classroom*, phone calls, and e-mail for both direct and indirect instruction. To a lesser extent, they also used various online quizzes, *Google Drive*, *Zoom*, and social networks, while they used the other tools to an even lesser extent. It is noted that teachers use a rather wide range of tools, which can be seen from the answers to the open-ended question, where they had the opportunity to state themselves if they used something that was not in the offered answers. The obtained result corresponds to previous studies that found that teachers are trained to use different tools (Akram et al. 2022; Guerrero-Ortiz et al. 2020; Mihajlović, Vulović, Maričić 2021).

When it comes to the application of teaching materials, both in the application of direct and indirect instruction, teachers indicated that they most often use text materials. PowerPoint, Prezi and other types of presentations, additional content and explanations, exercise materials, simulations, and charts, diagrams, and illustrations are somewhat more often used in direct instruction. Links to useful content are more often used in indirect instruction. They use video materials much more often than audio materials in both direct and indirect instruction. This also indicates that teachers use a wide range of teaching materials when applying both types of instruction in online mathematics classes, and that they use a slightly larger number of materials when applying direct instruction, probably because it is easier for them to apply, and they use it more frequently, which is also expected considering previous studies (Fakhrunisa et al. 2020; Guerrero-Ortiz et al. 2020).

CONCLUSION

The main contribution of this research is reflected in the examination of how teachers perceive the application of indirect versus direct instruction in online mathematics classes, as well as what factors could be influences associated with that perception. The general research hypothesis was confirmed. When it comes to practical implications, in accordance with the perception of the benefits and demands that teachers face in this domain, it will be possible to create appropriate educational support and programs for the further development of teachers' competencies in this area. Especially when taking into account the important factors that contribute to the use of indirect instruction in online mathematics classes, the results indicate that this presents a special kind of challenge for teachers and that they tend to apply it somewhat less often than in other subjects. The use of indirect instruction, as this research showed, would especially contribute to overcoming communication problems, which are reported as a frequent problem of online teaching. For this reason it is particularly important to work on providing appropriate material and technical support to teachers.

The examined socio-educational factors – levels of education and work experience – proved to be significantly related to the teachers' perception of direct and indirect instruction. Through future research, it would be useful to examine whether any other internal and external factors are related to teachers' perceptions (e.g. their personal characteristics, school resources) about the application of direct and indirect instruction in online mathematics teaching. Also, it would be significant to study the effects of potential educational programs that could be implemented with the aim of empowering teachers to overcome challenges and to more often apply indirect instructions in online mathematics classes, bearing in mind all the positive sides of this approach, which they themselves are aware of.

REFERENCES

Aaron, Herlost (2015): W. Aaron, P. Herlost, Teachers' perceptions of students' mathematical work while making conjectures: an examination of teacher discussions of an animated geometry classroom scenario, *International Journal of STEM Education*.

Alferi, Brooks, Aldrich, Tenenbaum (2011): L. Alferi, P. J. Brooks, N. J. Aldrich, H. R. Tenenbaum, Does discovery-based instruction enhance learning?, *Journal of Educational Psychology*, 103(1), 1–18.

Appana (2008): S. Appana, A Review of Benefits and Limitations of Online Learning in the Context of the Student, the Instructor and the Tenured Faculty, *International Journal on E-Learning*, 7(1), 207–213.

Aung, Khine (2020): Y. M. Aung, K. M. Khine, A study of the effects of direct instruction and indirect instruction on students' achievement in geometry, *J. Myanmar Acad. Arts Sci.*, XVIII, 9C.

Baysu, Ağırdağ (2019): G. Baysu, O. Ağırdağ, Turkey: Silencing ethnic inequalities under a carpet of nationalism shifting between secular and religious poles, In: P. S. (Eds.), *The Palgrave handbook of race and ethnic inequalities in education*, Palgrave Macmillan, 1073–1096.

Brunstein, Betts, Anderson (2009): A. Brunstein, S. Betts, J. R. Anderson, Practice Enables Successful Learning Under Minimal Guidance, *Journal of Educational Psychology*, 101(4), 790–802.

Cvjetićanin, Maričić (2022): S. Cvjetićanin, M. Maričić, Doprinos primene direktne u odnosu na indirektnu hands-on instrukciju na postignuća učenika u početnom obrazovanju u prirodnim naukama, *Inovacije u nastavi – časopis za savremenu nastavu*, 35(1), Belgrade: University of Belgrade, Teacher Education Faculty, 75–90.

Dean, Kuhn (2007): D. Dean Jr, D. Kuhn, Direct instruction vs. discovery: The long view, *Science Education*, 91(3), 384–397.

DeCaro, Rittle-Johnson (2012): M. S. DeCaro, B. Rittle-Johnson, Exploring mathematics problems prepares children to learn from instruction, *Journal of Experimental Child Psychology*, 113, 552–568.

Dignath, Veenman (2021): C. Dignath, M. V. Veenman, The role of direct strategy instruction and indirect activation of self-regulated learning – Evidence from classroom observation studies, *Educational Psychology Review*, 33(2), 489–533.

Eysnik, De Jong (2012): T. Eysnik, T. De Jong, Does Instructional Approach Matter? How Elaboration Plays a Crutial Role in Multimedia Learning, *Journal of the Learning Sciences*, 21(4), 583–625.

Fakhrunisa, Prabawanto (2020): F. Fakhrunisa, S. Prabawanto, Online Learning in COVID-19 Pandemic: An Investigation of Mathematics Teachers' Perception, *In 2020 The 4th International Conference on Education and E-Learning (ICEEL 2020)*, November 06–08, 2020, Yamanashi, Japan, NY, USA: ACM, New York.

Guerrero-Ortiz, Huincahue (2020): C. Guerrero-Ortiz, J. Huincahue, Mathematics teacher' perceptions and adaptations in developing online classes – ideas for teacher training, *Journal of Physics: Conference Series*, 1702.

Hmelo-Silver, Duncan, Chinn (2007): C. E. Hmelo-Silver, R. G. Duncan, C. A. Chinn, Scaffolding and Achivement in Problem-Based and Inquiry Learning: A Response to Kirschner, Sweller and Clark (2006), *Educational Psychologist*, 42(2), 99–107.

Hohlfeld, Ritzhaupt, Dawson, Wilson (2017): T. N. Hohlfeld, A. D. Ritzhaupt, K. Dawson, M. L. Wilson, An examination of seven years of technology integration in Florida schools: Through the lens of the levels of digital divide in schools, *Computers & Education*, 113, 135–161.

Jones, Southern, (2003): E. D. Jones, T. W. Southern, Balancing perspectives on mathematics instruction, *Focus on Exceptional Children*, 35(9).

Kalyuga, Chandler, Tuovinen, John (2001): S. Kalyuga, P. Chandler, J. Tuovinen, S. John, When Problem Solving Is Superior to Studying Worked Examples, *Journal of Educational Psychology*, 3, 579–588.

Kirschner, Sweller, Clark (2006): P. A. Kirschner, J. Sweller, R. E. Clark, Why Minimal Guidance During Instruction Does Not Work: An Analysus of the Failure of Constructivist, Discovery, Problem-Based, Experiential, and Inquiry-Based Teaching, *Educational psychologist*, 41(2), 75–86.

Kittell (1957): J. E. Kittell, An experimental study of the effect of external direction during learning on transfer and retention of principles, *Journal of Educational Psychology*, 48(7), 391–405.

Kopas-Vukašinović, Mihajlović, Miljković (2021): E. Kopas-Vukašinović, A. Mihajlović, A. Miljković, Stavovi studenata o kvalitetu nastave na daljinu, *Uzdanica*, XVIII/2, 19–33.

Kuhn (2007): D. Kuhn, Is direct instruction an answer to the right question?, *Educational psyhologist*, 42(2), 109–113.

Lee, Anderson (2013): H. S. Lee, J. R. Anderson, Student Learning:What Has Instruction Got to Do With It?, *The Annual Review of Psychology*, 64, 445–469.

Loibi, Rummel (2013): K. Loibi, N. Rummel, The impact of guidance during problem-solving prior to instruction on students' inventions and learning outcomes, *Instructional Science*, 42(3), 305–326.

Maričić, Cvjetićanin, Adamov, Ninković, Anđić (2022a): M. Maričić, S. Cvjetićanin, J. Adamov, S. O. Ninković, B. Anđić, How Do Direct and Indirect Hands-on Instructions Strengthened by the Self-Explanation Effect Promote Learning? Evidence from Motion Content, *Research in Science Education*, 1–21.

Maričić, Cvjetićanin, Andevski, Anđić (2022b): M. Maričić, S. Cvjetićanin, M. Andevski, B. Anđić, Effects of Withholding Answers Coupled with Physical Manipulation on Students' Learning of Magnetism-related Science Content, *Research in Science & Technological Education*.

Matlen, Klahr (2013): B. J. Matlen, D. Klahr, Sequential effects of high and low instructional guidance on children's acquisition of experimentation skills: Is it all in the timing?, *Instructional Science*, 41(3),

Mayer (2004): R. Mayer, Should there be a three-strikes rule against pure discovery learning?, *American Psychologist*, 59, 14–19.

Mihajlović, Vulović, Maričić (2021): A. Mihajlović, N. Vulović, S. Maričić, Teaching Mathematics During the Covid-19 Pandemic – Examining the Perceptions of Class Teachers and Mathematics Teachers, *Nauka, nastava, učenje u izmenjenom društvenom kontekstu*, Jagodina: Faculty of Education, 501–518.

Nifdi, N. I. (2022): *DI vs. di: The Term "Direct Instruction"*. Retrieved in September 2022 from https://www.nifdi.org/what-is-di/di-vs-di.html.

Oladayo, Oladayo (2012): O. T. Oladayo, C. E. Oladayo, Effects of Direct and Indirect Instructional Strategies on Students' Achivement in Mathematics, *African Research Review*, 6(4), 349–361.

Özdemir (2016): C. Özdemir, Equity in the Turkish education system: A multilevel analysis of social background influences on the mathematics performance of 15-year-old students, *European Educational Research Journal*, 15(2), 193–217.

Schmidt, Loyens, Van Gog, Paas (2007): H. G. Schmidt, S. M. Loyens, T. Van Gog, F. Paas, Problem-Based Learning is Compatible with Human Cognitive Architecture: Comentary on Kirschner, Sweller and Clark (2006), *Educational Psychologist*, 42(2), 91–97.

Singh, Thurman (2019): V. Singh, A. Thurman, How many ways can we define online learning? A systematic literature review of definitions of online learning, *American Journal of Distance Education*, 33(4), 289–306.

Stockard, Wood, Coughlin, Rasplica Khoury (2018): J. Stockard, T. W. Wood, C. Coughlin, C. Rasplica Khoury, The Effectiveness of Direct Instruction Curricula: A Meta-Analysis of a Half Century of Research, *Review of Educational Research*, 88(4), 479–507.

Trybus (2013): M. Trybus, Preparing for the future of education – Equipping students with 21st century skills: An interview with Dr. Robin Fogarty, *Delta Kappa Gamma Bulletin*, 80(1), 10–15.

Upu, Buhari (2014): H. Upu, B. Buhari, *Constructivism versus Cognitive Load Theory: In Search for an Effective Mathematics Teaching*. Retrived in September 2022 from https://arxiv.org/ftp/arxiv/papers/2108/2108.04796.pdf.

Warner, Kaur (2017): S. Warner, A. Kaur, The Perceptions of Teachers and Students on a 21st Century Mathematics Instructional Model, *International electronic journal of mathematics education*, 12(2), 193–215.

Yılmaz, Gülbağcı Dede, Sears, Yıldız Nielsen (2021): Z. Yılmaz, H. Gülbağcı Dede, R. Sears, S. Yıldız Nielsen, Are we all in this together?: mathematics teachers’ perspectives on equity in remote instruction during pandemic, *Educational Studies in Mathematics*.

Маријана М. Горјанац Ранитовић

Универзитет у Новом Саду, Педагошки факултет у Сомбору
Катедра за математику и методику наставе математике

Мирјана Т. Маричић

Катедра за природне науке и менаџмент у образовању

Миа Р. Марић

Катедра за друштвене науке

Александар В. Петојевић

Снежана С. Гордић

Катедра за математику и методику наставе математике

ПЕРЦЕПЦИЈЕ УЧИТЕЉА О ИНСТРУКТИВНОМ ВОЂЕЊУ У ОНЛАЈН-НАСТАВИ МАТЕМАТИКЕ

Резиме: Основни циљ овог истраживања представља испитивање начина на који учитељи опажају захтеве и допринос примене индиректне у односу на директну инструкцију. Поред тога, један од циљева је био и испитати однос социоедукативних варијабли: радна средина, ниво образовања и године радног искуства са начином на који учитељи опажају примену индиректне у односу на директну инструкцију у онлајн-настави математике. Утврђено је и да ли у поређењу са другим предметима, у настави математике учитељи чешће примењују одређену врсту инструкција, као и које наставне материјале и средства за комуникацију примењују приликом употребе директне и индиректне инструкције у онлајн-настави математике.

Ово истраживање је показало да учитељи у значајној мери увиђају позитивне стране примене индиректне у односу на директну инструкцију. Као највеће доприносе примене индиректне у односу на директну инструкцију, виде припрему ученика за самосталан рад и позитиван утицај на развој ученичких компетенција. Када се ради о захтевима примене индиректне у односу на директну инструкцију у онлајн-настави математике, учитељи у највећој мери оцењују да ова врста инструкција захтева веће материјалне и техничке ресурсе, више времена за припрему часа, те да је њена примена генерално комплекснија и да представља већи изазов за учитеље. Показало се да се перцепција карактеристика примене директне и индиректне инструкције у онлајн-настави математике донекле разликује, у зависности од тога да ли су у питању искуснији или мање искусни учитељи, као и тога колики је степен њиховог претходног образовања. Учитељи из градске и сеоске средине у подједнакој мери су свесни

доприноса и захтева примене индиректне у односу на директну инструкцију, те ово може имати позитивне импликације за наставну праксу.

Утврђено је да приближно половина учитеља, у поређењу са осталим предметима, чешће користи директну инструкцију у онлајн-настави математике, док око трећине учитеља подједнако примењује директну и индиректну инструкцију, као и у другим предметима. Свега око 14% учитеља навело је да чешће користи индиректну у односу на директну инструкцију у онлајн-настави математике.

Што се тиче примене средстава за комуникацију у онлајн-настави математике, учитељи су известили да и код примене директне и код примене индиректне инструкције претежно користе *Viber*, *Google Classroom*, разговор телефоном и имејл. Учитељи су известили да користе широк спектар наставних материјала код примене обе врсте инструкција у онлајн-настави математике.

Када је реч о практичним импликацијама, у складу са виђењем доприноса и захтева који се налазе пред учитељима у овом домену, биће омогућено и креирање одговарајуће подршке и програма за даљи професионални развој учитеља, посебно када се узме у обзир значај и улога коју приписују примени индиректних инструкција у онлајн-настави математике и резултат који говори о томе да за њих ово представља посебну врсту изазова, те да су склони да је примењују ипак нешто ређе него у осталим предметима. Употреба индиректних инструкција посебно би допринела превазилажењу проблема у комуникацији, који се наводи као чест недостатак онлајн-наставе.

Кључне речи: директна и индиректна инструкција, онлајн-настава математике, основно образовање, ставови учитеља.

Nenad R. Vulović
Aleksandra M. Mihajlović
Milan P. Milikić
University of Kragujevac
Faculty of Education in Jagodina
Department of Didactics and Methodology

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ACHIEVEMENTS OF YOUNGER PRIMARY SCHOOL STUDENTS IN MATHEMATICS COMPETITIONS DURING IHE COVID-19 PANDEMIC[1]

Abstract: Mathematics competitions represent a very important segment of educational support to gifted students and play a significant role in identifying, motivating and working with those mathematically gifted. The COVID-19 pandemic had a strong impact on all segments of the educational process, including the implementation of mathematics classes, regular and additional, as well as the organization and implementation of mathematics competitions. In this paper we wanted to examine whether the changed conditions in which regular and additional mathematics classes were implemented had an impact on the achievement of the best-performing math students. The aim of the research is to examine the adoption of advanced level mathematical content that students should have acquired at school in the conditions of the COVID-19 pandemic. The research sample consisted of 4,064 third-grade students (school year 2020/2021) and 3,824 fourth-grade students (school year 2021/2022). The research results indicate that students who should be able to solve advanced level tasks show insufficient practice in performing the four basic calculation operations, as well as insufficient adoption of different methods of solving advanced level tasks. By looking at the achievements of the same generation of students through two consecutive competition cycles, it can be seen that insufficiently adopted concepts in the third grade during the first year of the pandemic remained unexplained in the transition to a higher grade where they represent a problem for further advancement of students.

Keywords: mathematics competitions, younger primary school students, COVID-19 pandemic, student achievement.

INTRODUCTION

In 2020, the whole world experienced a global crisis, and many countries introduced extreme measures in reaction to the spread of the coronavirus disease.

[1] The paper is the result of research conducted within the bilateral project titled “Crises, Challenges and Current Education System” realised in collaboration between the Faculty of Education in Jagodina, University of Kragujevac (Serbia) and the Faculty of Philosophy, University of Montenegro (Montenegro) (2021–2023).

vlnenad@gmail.com

The COVID-19 pandemic led to the disruption of almost all aspects of social life, and educational systems around the world were not an exception (Mihajlović, Vulović, Maričić 2021). In March 2020, the Government of Serbia declared a state of emergency due to the COVID-19 crisis (Official Gazette 29/2020). Classes in the second semester of the academic year 2019/2020 were interrupted due to the outbreak of the pandemic and the declaration of a state of emergency (Official Gazette 30/2020) and everything was organized remotely. In school year 2020/2021, classes for students in the lower grades of primary school were realized according to a model in which students were divided into two groups. Students followed live classes in the classroom, but classes were shortened, and their number was reduced. However, the reduction in the number of classes did not apply to mathematics and mother tongue classes. On the other hand, due to the division of students into two groups and the additional workload of the teachers, there was not enough time for additional mathematics classes (either they were held rarely or not at all). In school year 2021/2022, all younger grade students returned to normal work mode.

The drastic changes due to the COVID-19 pandemic heavily influenced some crucial aspects of the organization of mathematics competitions around the world such as putting participants in the same controlled space in order to ensure equal opportunities and getting the jury together (to design tasks, control and supervise competition, and ensure proper marking) (Kenderov 2022). In order to avoid the introduced restrictions and survive, many mathematics competitions had to be organized online. In the Republic of Serbia, all mathematics competitions in the period from 2019 to 2022 were held, as planned, live but with increased epidemiological measures. Minimal shifts in the dates of the competition occurred only in the academic year 2019/2020 (Ognjanović, Hadži-Purić, Đukić 2020). Regardless of the fact that the dynamics of the competition were not disrupted, the question arose as to whether the changed conditions in which regular and additional mathematics classes were held had an impact on the best-performing math students' achievement. Some studies reported that changes in educational settings during the pandemic-affected school years had negative impacts on students' mathematical performance in general (Contini et al. 2022; Kuhfeld et al. 2022; Lewis et al. 2021; Moliner, Alegre 2022), and that lower grades were more negatively affected than higher grades (Asakawa, Ohtake 2022). Although some authors indicated that the high-achieving math students were less affected comparing to low- and average-achieving students (Schult et al. 2022), there were no studies that focused on investigating the effects of changed conditions on achievement of the best-performing math students in mathematics competitions.

With our research, we wanted to examine the effects of mathematics lessons implemented in changed conditions due to the COVID-19 pandemic on the achievements of younger primary school students in mathematics competitions.

THEORETICAL APPROACH TO THE PROBLEM

Mathematics competitions have a long tradition; they are organized in different forms, in different places and intended for different types of students. The first recorded mathematics competition dating back to 1885 was organized in Romania and included seventy students (Berinde 2004), eleven of whom received prizes (nine boys and two girls) (Kenderov 2022). In the following years, numerous countries started organizing mathematical competitions, considering them "as potentially rich opportunities for attracting young learners by proposing unusual non routine problems thus creating more opportunities for challenge they need and like" (Applebaum, Freiman 2013: 144).

Mathematical competitions represent a very important element of providing educational support to gifted students and play a significant role in the identification, motivation and support of gifted students in mathematics (Bicknell, Riley 2012; Toh 2015) and as such have a positive impact on the entire mathematics education development. Competitions provide an opportunity for students to explore new possibilities for doing mathematics that is not an integral part of the school subject of Mathematics. Such experiences allow students to apply the skills they have acquired in new situations and thus enrich their learning experience (Kenderov et al. 2009). Investigating the role of mathematics competitions in fostering students' interest in mathematics, Karnes and Riley (1996) point out that they can improve students' independent learning skills and autonomy.

Studies indicate that participation in mathematics competitions increase the likelihood that students will later have successful careers in STEM fields (Campbell, O'Connor-Petruso 2008; Steegh et al. 2019). Research shows that the first career orientations begin to form around the age of nine (Auger, Blackhurst, Wahl 2005) and continue to develop during later stages of schooling.

Regular teaching is largely based on enabling students to perform simple procedures, that is, it rarely teaches students to independently find solution methods or engage them in other mathematical processes (Lithner 2017). In a study analyzing the contents of mathematics textbooks in the USA, Australia, Canada, Finland, India, Ireland, Nepal, Scotland, Singapore, South Africa, Sweden and Tanzania, Jäder, Lithner, and Sidenwall (2015) found that 79% of tasks can be solved by applying given procedures, 13% of tasks require minor modifications of the presented solution models, and only 9% of tasks require students to construct procedures. On the other hand, solving difficult tasks that require students to have higher levels of reasoning not only generates better knowledge, but also cultivates skills for dealing with both mathematical and other types of problems (Kenderov 2022).

Mathematics competition tasks are designed to test the creativity, fluency and critical thinking of mathematically talented students. This population of students, who are the main focus of the existing literature when reporting on mathematics competitions (Rosolini 2011; Soifer 2012), usually expects to find among them exciting topics that they do not have the opportunity to get acquainted with in regular classes (Geogiev et al. 2008). This is the reason why the preparation of students for the competition has a significant educational impact, since by preparing for the competition, students' mathematical abilities are discovered and further developed (Kenderov 2022).

In the Republic of Serbia, the oldest, largest and best organized mathematics competitions for primary school students are organized by the Mathematical Society of Serbia (MSS). The first such competition at a republic-wide level was held in Belgrade in 1967 with the participation of 100 of the best eighth-grade students (Vulović 2016). Nowadays, it is estimated that annually close to 100,000 students from the third to the eighth grade of primary school participate in the initial levels of the competition (including the school and municipal level), while approximately thirty students reach the highest level of the national competition (Serbian Mathematical Olympiad) (Andrić et al. 2018). Students of the first and second grade of primary school are not included in mathematics competitions organized by MSS, while for the third and fourth grade primary school students, the highest level is the district competition.

RESEARCH METHODOLOGY

The research subject presented in this paper is the mathematics competitions of third and fourth grade primary school students. The problem we will look at is the achievements of the same generation of students in mathematics competitions in two consecutive school years. The competitions were organized by the Mathematical Society of Serbia and the Ministry of Education, Science and Technological Development of the Republic of Serbia. The research objective is to examine the adoption of advanced level mathematical content, which students should have acquired at school during the COVID-19 pandemic. Therefore, by analyzing the achievements of the same generation of students when they were in the third and fourth grades, we will see to what extent the knowledge was adopted during the pandemic. For the purpose of a better overview, we will base our analysis on the results of students at the municipal level of the competition. The reason for this is the degree of complexity of the tasks that are done in the municipal competition because they are the most accessible to the wider population of students. The municipal competitions taken into account were held in February 2021 and February 2022. The skills and knowledge that the students would have to demonstrate in the competition are known at the beginning of each school year and are available to all interested parties on the website of the Mathematical Society of Serbia.

The population of third grade students in the school year 2020/21 was 62,466, and the population of fourth grade students in the school year 2021/22 was 62,461 students. The competition was attended by 4,465 third grade students (7.15% of the total number of students) in the school year 2020/21, and in the school year 2021/22 there were 4,179 fourth grade students (6.69% of the total number of students).

The research sample which will be the basis for performing the analysis is given in Table 1.

Table 1. Research sample

School year	Grade	Sample size	% compared to the number of contestants	Gender	
				Male	Female
2020/21	3	4064	91.02%	2208 (54.33%)	1856 (45.67%)
2021/22	4	3824	91.51%	2101 (54.94%)	1723 (45.06%)

Research could not be conducted with the entire population since a certain number of schools did not submit students' points by tasks, but the total number only. The research instrument of both school years was a 5-task test compiled by the State Commission for the Competition of Students in Mathematics. The given tasks, solutions and method of evaluation are available at https://dms.rs/matematika-osnovne-skole/. The solutions to the tasks were evaluated partially according to a pre-defined evaluation key which all teachers who reviewed the tasks were familiar with regardless of the municipalities where the competition was held. The test time limit is 120 minutes. Data on student achievement were collected from the competition organizer's schools, MSS branches and Ministry branch units. Before the competition, the parents of all students signed an agreement to allow the processing of the results achieved by the students.

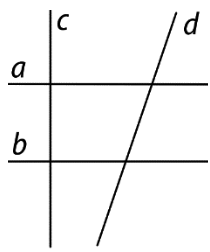
The collected data and numerical points were processed in the SPSS Statistic 20.00. Statistical measures and procedures that were used are: frequency, percentages, arithmetic mean and Mann–Whitney test.

RESULTS AND DISCUSSION

In our analysis, we will first focus on the achievements of third-grade students in a competition that was organized almost a year after the declaration of the pandemic. Teaching in the period from the beginning of the pandemic to the moment of the competition was mainly organized according to a combined model (in-school and online). It is important to note at the beginning of the analysis that the students who participate in the municipal competition are the best in mathematics in the communities they come from, because in order to come to the municipal competition, it is necessary to pass one level of selection: the school competition.

Based on the educational standards for the end of the first cycle of compulsory education, observing the mutual relationships of geometric objects is at the level of knowledge that belongs to the middle level of student achievement. The first task in the competition was aimed at observing the relationship between the lines in Figure 1. The students were asked to count how many lines are drawn in the figure and list which lines are parallel and which are normal.

Figure 1. Figure with task 1



A total of 48.23% of students did all three requirements correctly, while 5.44% of students did not do any part of the task correctly. In addition, 9.10% of the best third-grade students in the Republic of Serbia only knew how to count the lines that were given in the figure. The remaining students, in addition to listing straight lines, were able to spot, to the greatest extent, parallel lines (12.74%), while 8.76% of students listed both lines c and d as parallel. When specifying normal lines, 10.85% wrote down only one pair.

If we bear in mind that the knowledge required in this task is of fundamental importance for the further mathematical education of students, we can conclude that the level of their adoption is not satisfactory.

The second task was related to extracting the numbers of the first thousand according to a predetermined criterion. In this task, 17.62% of students did not score a single point, and the most common error in the task was writing down numbers that have the digit 2 in the place value of the hundreds, probably because the setting says that it is necessary to write down the numbers of the second hundred. In this task, 12.48% of students stopped working after writing down only one correct solution. Of the total number of students, 58.61% of them listed all 10 numbers in full, while 7.99% omitted one number when listing the numbers. The students' results in this task lead to the conclusion that students of this age need to insist more on tasks in which the solution is a set of numbers, as well as the necessity of emphasizing systematic answers in the students' work, so that they exhaust the entire set of solutions.

The third task was actually the only task in the competition in which the knowledge that is primarily acquired in additional mathematics classes is used. Magic squares appear only sporadically in mathematics textbooks, so the observation and acquisition of their properties is exclusively in these additional classes. The students' results on this task indicate that during the pandemic period in most schools, additional mathematics classes were either absent or held at a reduced capacity, as 45.79% of students did not know how to determine a single number in the magic square out of the ten required. Although the configuration of the numbers in the magic square is given in such a way that the students can easily determine the magic sum and then by a series of additions and subtractions in the range of zero to 100 determine the numbers that need to be written in the fields of the magic square, only 26.87% of the students managed to complete the task. The other students, in the majority of cases, made calculation errors after correctly determining two or three numbers, even though the required calculation was in the range of zero to 100.

The fourth task was the combinatorial type. Based on the given numbers, the students were supposed to compose one three-digit and one two-digit number whose sum or difference is equal to the given number. Although the first part of the requirement, determining sums, had four solutions, students were asked to provide only one solution. 28.05% of students could not solve any part of the task. 71.95% of the students did the part in which addends are determined, while 41.17% did the part in which the minuend and subtrahend were determined. The fact that more than half of the students failed to put together two numbers whose difference is given indicates that combinatorial problems were probably done to a lesser extent, but also that the students are not able to systematically look at all the possibilities for the solution of the task, even though their number in the specific task is small.

The worst performed task in the competition was the fifth task in which the students were asked to add two natural numbers according to a predetermined criterion and to determine their difference. The criteria in terms of which the students had to model the numbers were related to the sum and product of the digits of the number. These concepts can be mentioned informally in regular classes, while the actual application is more done in additional classes. The findings of this task support the conclusion from the third task about the inadequacy of additional classes, since 48.62% of the best third-grade students could not determine either the smallest three-digit number with the given sum of digits, or the largest three-digit number with the given product of digits. The works of other students show that determining a three-digit number with a given product of digits was a much more difficult problem than determining a three-digit number with a given sum of digits. 33.49% of students knew how to determine only the second number, while 17.89% of students managed to determine the first number as well. It should be noted that 1.38% of students had a problem and failed to calculate the difference between these two numbers.

The average number of students' points for each task for the third grade is given in Table 2.

Table 2. Average number of points for each task in the third grade

Task	1.	2.	3.	4.	5.
Average points	15.30	13.68	7.01	11.38	6.26

In this discussion, we will also look at the achievements of students in relation to gender. The average number of students' points in relation to gender is given in Table 3.

Table 3. Average number of points for each task in relation to gender

Tasks	1.	2.	3.	4.	5.
Boys	14.90	13.74	7.12	12.10	6.35
Girls	15.78	13.61	6.89	10.53	6.16

Student scores for each task do not have a normal distribution. Through statistical testing, we can conclude that there is a statistically significant difference in the achievements of boys and girls in the first and fourth task (Table 4).

Table 4. Mann–Whitney test results by tasks

Task	1.	2.	3.	4.	5.
U	1884775.5	2029922.5	2027914	1833510	1990136
p	0.000	0.564	0.546	0.000	0.087

Girls were better at noticing geometric relationships. There is a higher percentage of boys who did not score a single point on the task (6.43%) compared to girls (3.18%), but also a higher percentage of girls who scored the maximum number of points on the task (51.72%) compared to boys (45.29%).

In the combinatorial problem in the fourth task, boys did better, because 45.43% of them managed to solve the task completely, while for girls this percentage is 36.10%. In contrast, 24.68% of boys failed to solve any part of the task, while for girls this percentage is 30.93%.

The total number of points in the municipal competition in the third grade of boys (mean of 54.20 points) and girls (mean of 52.97 points) does not have a normal distribution and we may state that there is a small but statistically significant difference ($p = 0.049$) in the total achievements between them.

The results from the competition of the same generation of students in the fourth grade can illustrate the success of students in the second year of the pandemic, when students had lessons in schools almost all the time.

The first task in the fourth grade required students to compose expressions based on the given text. A large number of students (83.42%) managed to correctly compose and calculate the value of the composed expression. However, there is a significant number of those who: did not do any part of the task correctly (1.75%); incorrectly calculated the minuend and subtrahend (9.02%); correctly calculated both the minuend and subtrahend, but did not correctly calculate the value of the expression (5.65%). Such data indicate that a large number of students still have problems with performing basic arithmetic operations (multiplication, division and subtraction of two numbers), especially bearing in mind that the task was performed by students who were the best in mathematics in their communities.

Unlike the first task, the remaining tasks were a big problem for the students. In the second task, the students were asked to determine three unknown numbers based on the given conditions. Although the task could be solved in several ways, for example by using the longer method or by using the dependence of the sum on the change of addends, even 60.56% failed to score a single point in this task. Although 31.96% of the students did the task correctly, another 8.89% of the students went in the right direction towards the solution, but they made random mistakes in the calculation, thanks to which they did not reach the correct solution. The last data, in addition to confirming the conclusion from the first task, also indicate an unsatisfactory degree of processing of different methods of solving the task.

The problem of measuring time in the third task, although it represents the material already covered in previous grades, was the worst done task in the competition. The initial problem in the assigned task was to determine the number of days between two dates, and even 77.09% of the students failed to fulfill this requirement, and thus did not win any number of points in the task. 4.00% of students stopped after having this task done, while 11.85% of students determined how many seconds the clock would be late but not the time that the clock would show. 7.06% of students fully completed the task. As measurement and measures occupy a relatively small pool of mathematics lessons in the third and fourth grades, this task indicates that the students needed additional support to acquire this knowledge. In addition, this task was the only one in the competition in which a situation from a real environment was given, which is normally the most difficult problem type for students, so the results show that it is necessary to provide students with additional support for these types of tasks.

In the fifth task at the municipal competition for the third grade, it was observed that the students did not adopt the concept of the product of digits of a number to a satisfactory level. The same trend remained in the fourth grade as 34.23% of students failed to determine the digits used to write down the required numbers in the fourth task or to determine at least one of those numbers. About a third of the students (33.26%) managed to write down all the required numbers, but again a large percentage of students (8.55%) made mistakes when they had to add the ten obtained numbers. Bearing in mind that 32.51% of the students failed to write down all the numbers and that they made incidental mistakes in stating the required numbers, this indicates the need for a more systematic approach to the study of contents in which it is necessary to state the entire set of numbers.

The only geometric problem in the fourth grade was given in the fifth task, which was supposed to examine the extent to which the students are able to see the perimeter of a figure as a sum of adequate constituent parts, as well as to visually notice the same parts in a given picture. As many as 76.49% of the best-performing math students of the fourth grade could not see the components of a square and a rectangle from the picture, nor could they tell by how much the perimeter of the rectangle is longer than the perimeter of the square, so they could not continue with the task. 4.26% of them noticed that the rectangle consists of 10 sides of the square, but they could not continue beyond this statement, while 5.18% of them, in addition to the mentioned statement, also noticed that the perimeter of the rectangle is greater than the perimeter of the square by the length of 6 sides of the square. Only 14.07% of students were able to finish this task, and therefore to determine the required perimeters. Although perimeter-related contents were covered in the third grade, and students had time to deepen them, continuously poor acquisition of geometric concepts during the pandemic period was noticeable.

The average number of students' points for each task for the fourth grade is given in Table 5.

Table 5. Average number of points for each task in the fourth grade

Task	1.	2.	3.	4.	5.
Average points	17.77	6.95	3.38	8.99	4.11

In relation to gender, the average number of students' points is given in Table 6.

Table 6. Average number of points for each task in relation to gender

Tasks	1.	2.	3.	4.	5.
Boys	17.67	7.14	3.68	8.94	4.38
Girls	17.89	6.71	3.02	9.06	3.79

Student scores for each task do not have a normal distribution. Through statistical testing, we can conclude that there is a statistically significant difference in the achievements of boys and girls in the third and fifth tasks, in favor of boys (Table 7).

Table 7. Mann–Whitney test results by tasks

Task	1.	2.	3.	4.	5.
U	1782693	1762212	1722228	1796554	1743762,5
p	0.214	0.107	0.001	0.683	0.020

Unlike the third grade, when the girls achieved a significantly better result than the boys, a year of working with geometric content managed not only to compensate for the difference in their achievements, but also to make the boys achieve better results in the fourth grade. Moreover, we can see that girls achieved better results at the level of simple observation of relationships in a plane, while now, after a year, boys achieved better results at the level of applying geometric knowledge. Also, the success of boys is more evident in the area of measurements and measures for time, where 15.28% of boys did the task correctly, while the percentage of girls is 12.59%. Although the percentage of boys and girls who did parts of the task correctly is approximately the same, there is also a noticeable difference in the percentage of students who did not do any part of the task correctly (boys – 65.30%; girls – 68.60%).

Although there was a statistically significant difference in the overall achievements of girls and boys in the municipal competition in the third grade, in the fourth grade the difference in the overall achievements of boys (41.82) and girls (40.46) was larger ($p = 0.085$).

Studies show that gender differences in math achievement are not large at the beginning of schooling, but increase in later stages of education (Spelke 2005). Boys and girls at the preschool level show similar levels of mathematical literacy, but already at the level of the third grade of primary school, there are differences in achievement (Applebaum, Kondratieva, Freiman 2013; Cimpian et al. 2016). All of these can have a significant impact on later career choices in STEM fields (Hyde et al. 2008; Hyde, Mertz 2009). If we consider that the first career orientations are formed around the age of nine (Auger, Blackhurst, Wahl 2005) and that participation in math competitions can be associated with the development of a successful career in STEM fields (Campbell, O'Connor-Petruso 2008; Steegh et al. 2019), we believe that more attention should be paid to these differences.

CONCLUSION

Taking into consideration the level of considered competitions, the difficulty of the given tasks and the overall results achieved by the students, it can be concluded that the additional mathematics classes were either not sufficiently or satisfactorily implemented during the pandemic. Students who should be able to solve advanced level tasks show insufficient practice in performing the four basic calculation operations, and even with long-term practice, the students did not acquire a routine for solving calculations. Among the students, insufficient adoption of different methods of solving tasks is noticeable. Also, the students are only partially systematic in presenting their ideas, which resulted in the omission of certain parts of the solutions in the tasks of different areas. As we looked at the achievements of the same generation of students through two consecutive competition cycles, it is clearly noticeable that insufficiently adopted concepts in the third grade, in the first year of the pandemic, remained unexplained even when moving to a higher grade, where they represent a problem for the further advancement of students. This is especially noticeable in topics for which a small number of lessons were assigned.

The findings in the discussion section indicate that it is necessary to work more with students on tasks that are assigned in the context of real situations, since these types of tasks are done the worst, and this type of task is the most represented on all international student testing.

In presenting the competition results, we looked at the success of students in relation to gender. The results show that there were minimal differences in content adoption. Girls were better in the initial acquisition of basic geometric concepts, but the difference went in favor of boys with regard to the level of application in later studies. The boys were better in combinatorial problems, but also in the content for which a smaller number of classes are provided. However, one should not ignore the specific situation imposed by the pandemic during which the mathematics competitions were held. This implies that future research should address the examination of differences in student achievement in mathematics competitions in relation to gender.

REFERENCES

Andrić, Kadelburg, Marić, Mladenović (2018): В. Андрић, З. Каделбург, М. Марић, П. Младеновић, 70 година рада Друштва математичара Србије, *Настава математике*, LXIII, 1–2, 1–16.

Applebaum, Freiman (2013): M. Applebaum, V. Freiman, Attracting primary school children to mathematics: The case of a city mathematical marathon, *European Journal of Science and Mathematics Education*, 1(3), 144–157.

Applebaum, Kondratieva, Freiman (2013): M. Applebaum, M. Kondratieva, V. Freiman, Mathematics Competitions and Gender Issues: a Case of the Virtual Marathon, *Mathematics Competitions*, 26(1), 23–40.

Asakawa, Funio (2022): S. Asakawa, O. Funio, *Impact of COVID-19 School Closures on the Cognitive and Non-cognitive Skills of Elementary School Students.* Discussion papers 22075, Research Institute of Economy, Trade and Industry (RIETI), https://ideas.repec.org/p/eti/dpaper/22075.html.

Auger, Blackhurst, Wahl (2005): R. Auger, A. Blackhurst, K. H. Wahl, The development of elementary-aged children's career aspirations and expectations, *Professional School Counseling*, 8(4), 322–329.

Berinde (2004): V. Berinde, *Romania – The native country of International Mathematical Olympiads. A brief history of Romanian Mathematical Society*, CUB PRESS 22, Baia Mare.

Bicknell, Riley (2012): B. Bicknell, T. Riley, The role of competitions in a mathematics programme, *APEX: The New Zealand Journal of Gifted Education*, 17(1), 1–9.

Campbell, O'Connor-Petruso (2008): J. R. Campbell, S. A. O'Connor-Petruso, *National competitions help eradicate gender inequities in the gifted and talented*, Paper presented at the annual meeting of the European Council for High Ability, Prague, Czech Republic.

Cimpian, Lubienski, Timmer, Makowski, Miller (2016): J. R. Cimpian, S. T. Lubienski, J. D. Timmer, M. B. Makowski, E. K. Miller, Have gender gaps in math closed?

Achievement, teacher perceptions, and learning behaviors across two ECLS-K cohorts, *AERA Open*, 2(4), https://doi.org/10.1177/2332858416673617.

Contini, Di Tommaso, Muratori, Piazzalunga, Schiavon (2022): D. Contini, M. L. Di Tommaso, C. Muratori, D. Piazzalunga, L. Schiavon, Who Lost the Most? Mathematics Achievement during the COVID-19 Pandemic, *The B. E. Journal of Economic Analysis & Policy*, 22(2), 399–408, https://doi.org/10.1515/bejeap-2021-0447.

Decision on declaration a state of emergency, *RS Official Gazette*, No 29, March 15, 2020.

Geogiev, Ulovec, Mogensen, Mushkarov, Dimitrova, Sendova (2008): V. Geogiev, A. Ulovec, A. Mogensen, O. Mushkarov, N. Dimitrova, E. Sendova, *Meeting in mathematics*, Sofia: Comenius.

Hyde, Lindberg, Linn, Ellis, Willians (2008): J. S. Hyde, S. M. Lindberg, M. C. Linn, A. B. Ellis, C. C. Williams, Gender similarities characterize math performance, *Science*, 321(5888), 494–495.

Hyde, Mertz (2009): J. S. Hyde, J. E. Mertz, Gender, culture, and mathematics performance, *Proceedings of the National Academy of Sciences*, 106, 8801–8807.

Jäder, Lithner, Sidenvall (2015): J. Jäder, J. Lithner, J. Sidenvall, A cross-national textbook analysis with a focus on mathematical reasoning – The opportunities to learn, In: J. Jäder, *Elevers möjligheter till lärande av matematiska resonemang*, Licentiate thesis, Linköping University.

Karnes, Riley (1996): F. A. Karnes, T. L. Riley, Competitions: Developing and Nurturing Talents, *Gifted Child Today Magazine*, 19(2), 14–15.

Kenderov (2022): P. S. Kenderov, Mathematics competitions: an integral part of the educational process, *ZDM Mathematics Education*, 54, 983–996, https://doi.org/10.1007/s11858-022-01348-4.

Kenderov, Rejali, Bartolini-Bussi, Pandelieva, Richter, Mschietto, Kadijevich, Taylor (2009): P. Kenderov, A. Rejali, M. G. Bartolini-Bussi, V. Pandelieva, K. Richter, M. Mschietto, D. Kadijevich, P. Taylor, Challenges Beyond the Classroom – Sources and Organizational Issues, In: E. Barbeau, P. Taylor (Eds.), *Challenging mathematics in and beyond the classroom*, The 16th ICMI Study: New ICMI Study Series, Springer, 53–96.

Kuhfeld, Soland, Lewis, Ruzek, Johnson (2022): M. Kuhfeld, J. Soland, K. Lewis, E. Ruzek, A. Johnson, The COVID-19 School Year: Learning and Recovery Across 2020–2021, *AERA Open*, 8, https://doi.org/10.1177/23328584221099306.

Lewis, Kuhfeld, Ruzek, McEachin (2021): K. Lewis, M. Kuhfeld, E. Ruzek, A. McEachin, *Learning during COVID-19: Reading and math achievement in the 2020–21 school year*, NWEA. Retrieved in October 2022 from https://www.nwea.org/content/uploads/2021/07/Learning-during-COVID-19-Reading-and-math-achievement-in-the-2020-2021-school-year.research-brief-1.pdf.

Lithner (2017): J. Lithner, Principles for designing mathematical tasks that enhance imitative and creative reasoning, *ZDM mathematics Education*, 49, 937–949.

Mihajlović, Vulović, Maričić (2021): A. Mihajlović, N. Vulović, S. Maričić, Teaching Mathematics During the COVID-19 Pandemic – Examining the Perceptions of Class Teachers and Mathematics Teachers, *Proceedings of the International Scientific Conference "Science, teaching, learning in a changed social context"*, 501–518, Užice: University in Kragujevac, Faculty of Education in Uzice.

Moliner, Alegre (2022): L. Moliner, F. Alegre, COVID-19 Restrictions and Its Influence on Students' Mathematics Achievement in Spain, *Education Sciences*, 12(2), 105, https://doi.org/ 10.3390/educsci12020105.

Official Gazette of the Republic of Serbia, 30/2020.

Ognjanović, Hadži-Purić, Đukić (2020): С. Огњановић, Ј. Хаџи-Пурић, Д. Ђукић, Такмичења из математике и информатике у Србији 2020. године, *Настава математике*, 65(3–4), 113–118.

Rosolini (2011): G. Rosolini, The Italian Team Competition, *Mathematics Competitions*, 24(2), 14–22.

Schult, Mahler, Fauth, Lindner (2022): J. Schult, N. Mahler, B. Fauth, M. A. Lindner, Did students learn less during the COVID-19 pandemic? Reading and mathematics competencies before and after the first pandemic wave, *School Effectiveness and School Improvement*, 33(4), 544-563, DOI: 10.1080/09243453.2022.2061014.

Soifer (2012): A. Soifer, The Goal of Mathematics Education, including Competitions, is to let Student touch "Real" Mathematics: We Ought to Build that Bridge, *Mathematics Competitions*, 25(2), 47–67.

Spelke (2005): E. S. Spelke, Sex Differences in Intrinsic Aptitude for Mathematics and Science?: A Critical Review, *American Psychologist*, 60(9), 950–958, https://doi.org/10.1037/0003-066X.60.9.950.

Steegh, Höffler, Keller, Parchmann (2019): A. M. Steegh, T. N. Höffler, M. M. Keller, I. Parchmann, Gender differences in mathematics and science competitions: A systematic review, *Journal of Research in Science Teaching*, 56, 1431–1460.

Toh (2015): T. L. Toh, The roles of mathematics competition in Singapore mathematics education, In: S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education (ICME-12)*, Cham: Springer International Publishing, 640–656.

Vulović (2016): Н. Вуловић, Припрема ученика за математичка такмичења у разредној и предметној настави, *Узданица*, XIII(1), 71–86, ISSN 1451-673X.

Ненад Р. Вуловић

Александра М. Михајловић

Милан П. Миликић

Универзитет у Крагујевцу
Факултет педагошких наука у Јагодини
Катедра за дидактичко-методичке науке

ПОСТИГНУЋА УЧЕНИКА МЛАЂИХ РАЗРЕДА ОСНОВНЕ ШКОЛЕ НА МАТЕМАТИЧКИМ ТАКМИЧЕЊИМА ТОКОМ ПАНДЕМИЈЕ КОВИДА 19

Резиме: Математичка такмичења представљају веома важан сегмент пружања образовне подршке даровитим ученицима и играју значајну улогу у идентификацији, мотивацији и раду са математички даровитима. Пандемија ковида 19 је утицала снажно на све сегменте образовног процеса, па и на реализацију наставе математике, редовне и додатне, као и на организацију и реализацију математичких такмичења. У овом раду желели смо да испитамо да ли су измењени услови у којима су реализоване редовна и додатна настава математике имали утицај на постигнућа најбољих ученика. Циљ истраживања је испитивање усвојености математичких садржаја напредног нивоа које је требало да ученици стекну у школи у условима пандемије ковида 19. Узорак истраживања чинило је 4064 ученика трећег разреда (школске 2020/2021. године) и 3824 ученика четвртог разреда (школске 2021/2022. године). Резултати истраживања указују да ученици који би требало да решавају задатке напредног нивоа показују недовољну увежбаност извођења четири основне рачунске операције, као и недовољну усвојеност различитих метода решавања задатака напредног нивоа. Сагледавањем постигнућа исте генерације ученика кроз два узастопна такмичарска циклуса, уочава се да су недовољно усвојени концепти у трећем разреду, у првој години пандемије, остали неразјашњени и преласком у виши разред где они представљају проблем за даље напредовање ученика.

Кључне речи: математичка такмичења, млађи разреди основне школе, пандемија ковида 19, постигнућа ученика.

Jana Slezáková
Palacký University in Olomouc
Faculty of Science
Department of Experimental Physics,
Section of Pedagogical training
Czech Republic

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PEDAGOGICAL TRAINING FROM THE PERSPECTIVE OF STUDENTS – FUTURE TEACHERS AT PALACKÝ UNIVERSITY OLOMOUC

Abstract: This paper discusses the issue of perception and evaluation of the first continuous pedagogical training from the perspective of students as future science and mathematics teachers. In the introduction, attention is paid to the definition of pedagogical training from the perspective of selected authors. The following is an overview and a description of the skills of future teachers, which are key during the first continuous pedagogical training. Subsequently, the following section presents the organizational structure of the first training which is applied at the Faculty of Science at Palacký University Olomouc, Czech Republic. Further, for the purposes of the empirical research, the author drew up a survey with questions addressed to future science and mathematics teachers. Based on that, the data were analysed and transformed into results. The main intention of the survey was to investigate how the students appraise their preparation for their practical training and the training itself. Pedagogical training shows the need for the development of digital competencies.

Lastly, the final part of the paper discusses the shortcomings of their practical training, which are perceived by undergraduates as future teachers. Additionally, it presents suggestions and recommendations for how to effectively improve the organizational system of these practical teacher trainings.

Keywords: pedagogical training, teaching competencies, tandem teaching.

INTRODUCTION

The reality of today's education is increasingly focused on a constructivist approach to teaching aimed at individualizing the teaching process in order to put forward transmissive methods that suppress the development of the student's personality and are focused on the verbal monological concept of teaching. The education system is focused on the student's self-education, in which the teacher becomes a facilitator of the student's learning and thus teaches the student to work with new information, motivates him to link new information with previously acquired knowledge, and tries to apply and further interpret the student's information.

jana.slezakova@upol.cz

Through this experience, the student consolidates key competencies. An important task of the teacher is to transfer this information into the form of knowledge, skills, habits and attitudes that the student achieves during education. The basics of professional skills are acquired by future teachers in undergraduate training within general didactics and subject didactics.

The aim of our study was to determine the readiness of students of science and mathematics teaching for their first continuous teaching training. We wanted to determine in the form of a questionnaire survey how they evaluate this first training. We also tried to find out if they are able to prepare and implement a lesson independently. Furthermore, we try to show whether students are able to solve professional problems promptly. Furthermore, we wanted to know if the students are sufficiently prepared to work with pupils with special educational needs. And finally, we wanted to find out to what extent they are able to use digital technologies in the implementation of their own lessons.

THEORETICAL BACKGROUND

Pedagogical training as a part of the study of all teaching disciplines is an integral and necessary form of teacher preparation. The goal of pedagogical training is to connect theoretical education with the possibility of practical application of acquired knowledge. Pedagogical trainings help future teachers to integrate knowledge from general didactics directly into practice. As part of the preparation of future teachers, pedagogical trainings should serve to consolidate relevant pedagogical competencies. Trainings should develop social, communicative and interpersonal skills. They should also teach students self-reflection.

Pedagogical training can be defined in different ways. For example, Buchberger and Busch (1988: 90) define pedagogical training as the acquisition of skills directly related to the teaching process to encourage the ability and willingness to actively apply theory in practice. It is part of the study, influenced by experienced teachers in the school responsible for practical training.

Vonk (1985: 135) defines pedagogical training as an opportunity to learn, specifically through learning situations for future teachers in teacher education that are systematically confronted with possible practice, specific teaching activities and classroom management at school, led by special tutors and practicing teachers.

Šimoník (2005: 49) understands pedagogical training as an inseparable part of undergraduate teacher preparation, which is only a stage in the practical training of teachers, because it is not possible to practice everything we expect from a teacher. He points out the need to connect pedagogical training with theory. Šimoník states that pedagogical training should be a discipline integrating the theoretical and practical components of teacher preparation.

Průcha, Walterová, Mareš (2001: 258) characterize pedagogical training as a part of the practical preparation of teachers and educators at faculties preparing teachers. The main goals of the training include: to combine the theory and practice of all components of higher education, to introduce the future teacher to the conditions of the real school environment and to practice him in the activities of the teaching profession.

According to Nezvalová (2007: 8), the indicated definitions show that pedagogical training is understood as an opportunity for students to use their theoretical knowledge, verify their teaching skills and, based on observing the activities of experienced teachers and reflection of their own activities, to create individual teaching concepts.

APPROACH TO CREATING PROFESSIONAL SKILLS OF TEACHING STUDENTS OF THE FACULTY OF SCIENCE OF THE PALACKÝ UNIVERSITY IN OLOMOUC, CZECH REPUBLIC

Pedagogical training strongly influences students-prospective teachers in their decision-making and strengthens or weakens their decision to educate future generations. In general, trainings form an important part of educational preparation, as they connect the theoretical teaching of branch didactics with the practical. Related to this is the need for feedback from future teachers and their suggestions for improving the organization of pedagogical trainings.

Department of Pedagogical Preparation of the Faculty of Science, Palacký University in Olomouc provides pedagogical trainings for students of science and mathematics teaching. As part of their study programs, future teachers complete the first and second continuous pedagogical trainings. These trainings provide an opportunity to get to know the educational activities of the school as a whole. Students acquire skills that are directly related to the teaching process. During these trainings, they develop follow-up skills that we consider key during the first teaching training.

These include the following:

1. Planning and preparation for the lesson
2. Realization of lessons
3. Lesson management
4. Classroom climate
5. Evaluation of pupils’ results
6. Reflection of own activity and evaluation

Planning and preparation have a clear intention and goal, where the educational content and methods correspond to the educational needs and abilities of the pupils and the required outcomes. The lesson must be structured in such a way that it always builds on the previously acquired concepts in an appropriate way and creates preconditions for clarification and understanding of subsequent and related concepts. In the implementation phase of the lesson, the future teacher in the role of facilitator of student learning presents the planned content using appropriate organizational forms and teaching methods that will lead to the achievement of the planned learning objectives. The future teacher develops skills leading to the successful achievement of set goals. The student of teaching is already able to effectively monitor the results of their activities, correct their procedures, monitor students' activities and provide feedback.

With his attitudes and actions, the practicing student creates a positive classroom climate and corrects any inappropriate behavior of some students. As part of the reflection, he should be able to identify those sites that need further improvement and associated development.

Future teachers learn to apply all these skills in a specific situation at school in the presence of experienced teachers.

ORGANIZATION OF THE FIRST CONTINUOUS PEDAGOGICAL TRAINING AT THE FACULTY OF SCIENCE, PALACKÝ UNIVERSITY IN OLOMOUC

Pedagogical training is a part of the mandatory preparation of future science and mathematics teachers. In the bachelor's study program, students of teaching programs complete only theoretical preparation through the compulsory subject General and school didactics. In the follow-up master's study program, students have theoretical subjects supplemented by a mandatory first and second continuous teaching training. Both types of trainings are implemented at so called faculty schools that cooperate with the Faculty of Science of Palacký University Olomouc. These schools enable students to connect theoretical knowledge with practical experience.

Continuous pedagogical training is realized in the first year of the follow-up master's study, in the summer semester lasting 3 weeks. A student who intends to perform a continuous teaching practice submits an application. All documentation (date of training, list of students registered for pedagogical training, location of students at individual schools, information for students on the course of pedagogical training, forms) is located on the website of the Department of Pedagogical Training (cpp.upol.cz). The staff of this Department will provide students with the necessary recommendations at the information meeting. The student does not provide the training independently, but on the basis of his application the Department staff perform the necessary administrative activities (communication with the school management, list of students conducting continuous teaching training at school, information on continuous teaching training, requirements for continuous teaching training, agreement for the head teacher training). The student's teaching activity at the relevant school is evaluated by the head teacher of the school on the appropriate form.

During the pedagogical training of future teachers, the teaching student gets acquainted with the basic theoretical pedagogical and didactic knowledge and skills, which he then applies in his activities in a real school environment, thus creating his initial individual professional skills and attitudes under the guidance of experienced schoolteachers. Within the pedagogical training, emphasis is placed on the development of the teacher's personality. Pedagogical training and theory according to (Dytrtová, Krhutová 2009:40) proves that any ideal content of education is in itself an indifferent phenomenon, and the driving force is the teacher's personality and the methods of education that the teacher uses.

QUESTIONNAIRE SURVEY METHODS

How do future science and mathematics teachers, who have undergone only theoretical subjects in their undergraduate preparation, perceive their first continuous pedagogical training?

In our study, we focused on quantitatively oriented research. We used the questionnaire survey method (Gavora 2010: 67). Our goal was to create a simple questionnaire that would not inconvenience the respondents too much. This is because the students filled out the questionnaire in written form immediately after the end of their training. At the same time, we did not want it to address the teaching of other subjects. We were also aware that the questionnaire was filled out by students across science disciplines, so it was not possible to address them at the same time.

A total of 50 first-year follow-up students of science and mathematics took part in the questionnaire survey, of which 44% were women and 56% were men. Empirical methods focused on proportional stratified selection – Chráska (Chráska 2003: 35) were chosen for the survey. A questionnaire was presented to students in March 2021 (immediately after the end of the first continuous pedagogical training), which aimed to find out how future teachers perceive their readiness for the first continuous pedagogical training, whether this training met their expectations, what they see as the benefit of this training and what suggestions they have for possible improvements to the system of pedagogical training.

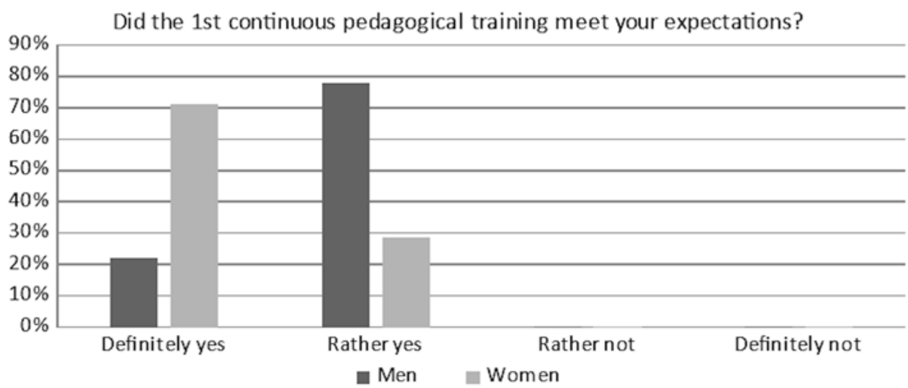
We created a group of pre-prepared and carefully formulated questions, which we have tried to arrange thoughtfully and to which the interviewed person answers in writing. At the same time, we realized that the obtained data require careful interpretation. This is to avoid subjective judgments.

The tested group consisted of students who have had a modified study program since the academic year 2019/2020 and who have not completed assistant or listening pedagogical training within their study program.

RESULTS OF THE QUESTIONNAIRE SURVEY

The introductory question asked whether the first continuous training met all the expectations of future science and mathematics teachers. In both groups of women/men examined, only answers *yes* or *rather yes* appeared. In the case of women, the answer was *definitely yes*, in the case of men *rather yes*. However, none of the respondents indicated the answer *rather no* or even *definitely no*.

Graph 1. Percentage expression of the fulfillment of expectations of future teachers for their first continuous pedagogical training



The second question asked what the biggest problems future teachers had in their first training and what surprised them the most during the training. Both groups of men/women mentioned the timing and organization of the teaching unit to the same extent, especially the correct estimation of the pace of interpretation. There were also answers regarding indiscipline in teaching, time-consuming preparation, working with pupils with special educational needs, prompt solutions of professional problems and graphic expression on the blackboard.

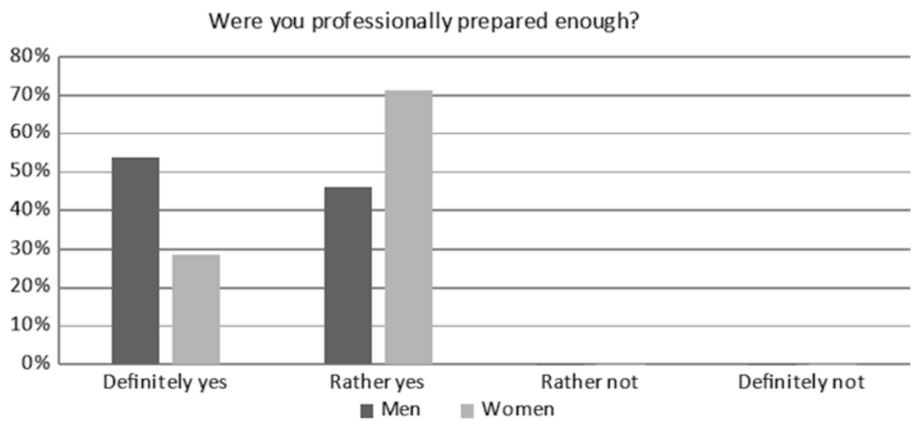
In the third question, students should state what they see as the benefit of their first continuous teaching training. Again, two types of responses appeared in both groups. The first was the answer that pedagogical training allows them to directly apply the acquired theory in practice. The second most common answer was to realize that it is a good experience that is a necessary part of preparation for future careers. The main ideas in the women's answers were that in the training they made sure that they "really wanted to pursue the teaching profession", that they "stood in front of the class for the first time", that they "only realized during the training what the teacher's job entails", etc. The men then mostly answered that they literally wanted to see whether they would enjoy this profession and wanted to get acquainted with the school environment from the teacher's point of view, etc.

The fourth question asked whether the students at the practicing school encountered something they were not prepared for during the first three years of study. Both groups of men and women responded "working with pupils with special educational needs". According to RVP G (MŠMT 2021), pupils with special educational needs are considered to be pupils with disabilities and pupils with social disadvantages. Diagnosing such pupils and creating conditions for their education is an extremely demanding activity for a teacher, which is carried out in cooperation with a pedagogical-psychological counseling center, a special pedagogical center, or a special pedagogue or a psychologist.

The fifth question focused on gauging the training of future teachers from a professional point of view. Both groups felt sufficiently trained. It is worth noting that the women were more inclined to answer *yes* to the question "Were you professionally prepared enough?", but men chose the first answer, *definitely yes*.

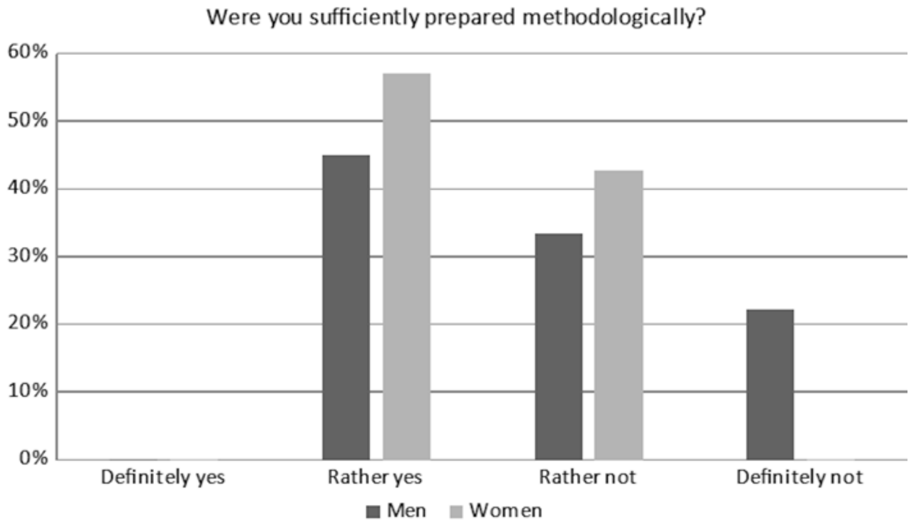
In their study, Pinheiro and Zaidan (2022: 447) discussed the importance of theoretical preparation, including the content evaluation of professional subjects, for the quality of teacher training.

Graph 2. Percentage expression of training of future teachers



The penultimate question examined the extent to which students are methodologically prepared. None of the respondents to the question "Were you sufficiently prepared from a methodological point of view?" answered unequivocally positive (*definitely yes*), the answers *rather yes* and *rather no* were almost balanced. *Definitely no* answers appeared in men.

Graph 3. Percentage expression of future teachers in terms of methodology



The last question gave room for suggestions for improving the system of pedagogical trainings. 31% of respondents would welcome longer training, 44% suggest strengthening the preparation of future teachers for listening and assistant training during their bachelor's studies, 50% of students said that they would like to focus on forms of teaching or teacher versus student communication within subject didactics, and 37% of respondents are interested in starting with tandem teaching during the first exposure to the school environment.

According to R. Dofková (2019: 12), tandem teaching is one of the less common forms of teaching through the specific experience of a practical teacher. Dofková states that this form of teaching leads to more efficient teaching hours, as it takes place in the presence of two or more teachers. It follows from the above that in the school environment there is also space for self-observation of the work of experienced teachers. Another aspect is the joint implementation of group teaching, in which the student – future teacher and head teacher can share activities and complement each other. All these aspects will enable future teachers to create their own idea of real events in school practice and at the same time the concept of their own quality teaching.

DISCUSSION

It can be stated clearly that the first continuous pedagogical training met the expectations of students. Although unexpected problems arose that they were unprepared for, they were able to deal with them properly. The training was also beneficial in that some students only clarified during it whether or not they want to devote themselves to pedagogical work. The respondents evaluated their own professional erudition very positively, but from a methodological point of view they themselves perceived significant shortcomings.

The study (Skafa, Evseeva, Abramenkova, Goncharova 2021: 212) describing the system of preparation of future teachers at the Donetsk National University also dealt with the improvement of the preparation of future mathematics teachers. It was an implementation of heuristic activities. The research results pointed not only to the importance of mastering various methods and forms of work, but also to the often-neglected methodical preparation.

An important factor that helps future teachers to better adapt in the school environment is also the development of critical thinking. Cobo-Huesa, Abril and Ariza (2022: 360203) studied the preparation of future primary education teachers. In their study, they proposed recommendations necessary for the educational preparation of future teachers. One of the factors that should not be neglected in education is, for example, preparation and planning for the lesson. This deficiency was also found in our survey.

Here is an overview of issues that appeared in the questionnaire responses repeatedly:

– are unable to properly structure the lesson with regard to the subject matter, teaching schedule, pace of interpretation
– are not able to immediately address professional issues from students
– are unable to deal promptly and correctly with disciplinary problems
– do not have sufficient ability to work with gifted pupils and pupils with special educational needs
– show shortcomings in the choice of appropriate teaching methods

The questionnaire survey also brought several stimulating suggestions for improving the concept of teaching training: inclusion of listening and assistant training during bachelor's studies, enabling tandem teaching in training, extension of continuous training, training of teacher-student communication and practical training of various forms and methods of teaching within subject didactics.

CONCLUSION

Based on the results obtained from the questionnaire survey, the importance of the implementation of pedagogical trainings in the undergraduate training of future teachers at the Faculty of Science, Palacký University Olomouc is confirmed.

Pedagogical trainings should fulfill three important tasks. Training should help science and mathematics teaching students to get to know the school and the school environment. Trainings should relieve students of fear during communication. Training should integrate the knowledge gained by studying at the faculty with the reality of school life. It follows that future teachers should observe school environments. Subsequently, they should move to their positions as educators. They should be able to confront this initial knowledge with their possibilities, motivation and perspective. It also follows that the absence of listening and observation practice in the bachelor's study program hinders awareness of the teacher's position in education.

In our research, we focused on the evaluation of pedagogical trainings of future science and mathematics teachers. We tried to capture the importance of pedagogical skills of future teachers. It is also important to realize that education is constantly changing. The educational process is becoming more and more interactive. Therefore, targeted preparation of future teachers for mobile education is also important. Sharafeeva (2022: 31) also dealt with this issue in her research.

Pedagogical trainings in general allow students to connect theoretical knowledge with a specific situation in the school; they provide a reflection of all activities not only from the position of the teacher but also the student. It develops the ethics of social communication with all participants in the educational process. Trainings strengthen competencies for planning, managing and diagnosing educational activities.

Our research also pointed to a lack of preparation for working with pupils with special educational needs. Future teachers should be better prepared to work with children with disabilities and health disadvantages. Studies dealing with the problems of teachers who taught children with disabilities were published by Berikkhanova (202: 675). The results of their pedagogical research confirmed the importance of the adaptation of future teachers in an inclusive environment.

In the field of digital competencies, it is necessary to focus on the transfer of knowledge in the field of information and communication technologies into the process of teaching science and mathematics, specifically the use of various software for e.g., validation of results, possible procedures for solving specific examples and visualization of specific issues. The use of mathematical software in science subjects enables students (future teachers) to better understand the current educational context, develop their sense of imagination and learn this way of thinking.

Everyday work in the educational environment helps to better develop pedagogical thinking. The trainings integrate all components of the university preparation of future teachers and thus form overall pedagogical competence. Teaching practice is one of the important basic pillars that will prepare future teachers for a very demanding but enriching professional career.

REFERENCES

Berikkhanova, Ospanova, Yermenova, Zharmukhametova, Sultanova, Urazbaeva (2021): G. Berikkhanova, B. Ospanova, B. Yermenova, R. Zharmukhametova, N. Sultanova, A. Urazbaeva, The Effectiveness of the Training Model of the Future Teacher in Conditions of Inclusive Education, *International Journal of Education and Practice*, IX/4, 670–686.

Buchberger, Busch (1988): F. Buchberger, F. W. Busch, The role of Practical Element in Initial Teacher Education, *European Journal of Teacher Education*, XI/ 2–3, 89–91.

Cobo-Huesa, Abril, Ariza (2022): C. Cobo-Huesa, A. M. Abril, M. R. Ariza, Pedagogical content knowledge for teaching of nature of science and critical thinking in initial primary teacher education, *Revista Eureka sobre Enseñanza y Divulgación de las Ciencias*, XIX/3, Universidad de Cádiz, 360201–360215.

Dofková (2019): R. Dofková, Tandemová výuka v matematice jako instrument reflektivního modelu vzdělávání, *e-Pedagogium*, XIX/1, UP Olomouc, 7–13.

Dytrtová, Krhutová, (2009): R. Dytrtová, M. Krhutová, *Učitel-příprava na profesi*, Grada Publishing: Praha.

Gavora (2010): P. Gavora, *Úvod do pedagogického výzkumu*, Paido: Brno.

Chráska (2003): M. Chráska, *Úvod do výzkumu v pedagogice*, UP Olomouc: Olomouc.

MŠMT (2021): *Rámcový vzdělávací program pro gymnaziální vzdělávání*. Retrieved in June 2022 from https://www.edu.cz/rvp-ramcove-vzdelavaci-programy/ramcove-vzdelavaci-programy-pro-gymnazia-rvp-g/

Nezvalová (2007): D. Nezvalová, *Pedagogická praxe v počáteční přípravě učitelů přírodovědných předmětů a matematiky pro střední školy*. Retrieved in May 2020 from http://esfmoduly.upol.cz/

Pinheiro, Zaidan (2022): N. V. Pinheiro, S. Zaidan, Learning Evaluation in Mathematics Teaching Degree and the Possible Implications for Teacher Training, *The Mathematics Enthusiast*, XIX/2, Montana, 442–469.

Průcha, Walterová, Mareš (2001): J. Průcha, E. Walterová, J. Mareš, *Pedagogický slovník*, Praha: Portál.

Sharafeeva (2022): L. Sharafeeva, A Model of Future Mathematics Teachers' Preparedness to Organize Mobile Learning for Schoolchildren, *Journal of Curriculum and Teaching*, XI/3, Special Issue, 30–37.

Skafa, Evseeva, Abramenkova, Goncharova (2021): E. I. Skafa, E. G. Evseeva, Y. V. Abramenkova, I. V. Goncharova, The system of training a new generation of mathematics teachers based on project and heuristic activities, *Perspectives of Science & Education*, LIII/5, 208–222.

Šimoník (2005): O. Šimoník, Pedagogická praxe – připomínky k jejich organizaci a průběhu, náměty na jejich zefektivnění, In: *Pedagogická praxe a profesní rozvoj studentů*, Brno: Masarykova Univerzita Brno, 48–53.

Vonk (1983): J. H. C. Vonk, Problems of Beginning Teacher, *European Journal of Teacher Education*, VI/2, 133–150.

Јана Слезакова

Универзитет Палацки у Оломоуцу
Факултет природних наука
Одсек за експерименталну физику, Одељење за педагошку праксу
Чешка Република

ПЕДАГОШКА ПРАКСА ИЗ ПЕРСПЕКТИВЕ СТУДЕНАТА – БУДУЋИХ УЧИТЕЉА СА УНИВЕРЗИТЕТА ПАЛАЦКИ У ОЛОМОУЦУ

Резиме: Рад се бави питањем перцепције и евалуације прве континуиране педагошке праксе из угла студената, будућих наставника природних наука и математике. Најпре је дат преглед и опис компетенција будућих наставника, које су кључне у оквиру прве континуиране праксе. У другом делу представљена је организациона структура прве праксе на Факултету природних наука Универзитета у Оломоуцу у Чешкој Републици. Циљ истраживања је био да се испита како студенти процењују сопствену припремљеност за праксу и саму праксу.

Кључне речи: педагошка пракса, компетенције наставника, извођење наставе у пару.

Aleksandar Z. Milenković
University of Kragujevac
Faculty of Science
Department of Mathematics and Informatics
Kragujevac, Serbia

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ALGEBRAIC STRUCTURES BY CREATING MIND MAPS WITH STUDENTS GIFTED IN MATHEMATICS

Abstract: Working with mathematically gifted students is the subject of many studies. In the literature, one can find various examples of the positive impact of the use of mind maps on learning by understanding and connecting concepts in appropriate schemes, but the impact of creating mind maps on the achievements of students gifted in mathematics has not been sufficiently researched. Having that in mind and that algebraic structures represent a teaching topic in which it is necessary for students to have adequate theoretical knowledge about these structures and relations between them, this method was implemented in order for students to connect the proper concepts in a scheme. For that purpose, (two) mathematics classes of systematization are conducted for the teaching topic on Algebraic structures in order for students to create two mind maps each (one for algebraic structures with one and with two binary operations and another for homomorphism). The effects of this approach to the systematization classes were examined by analyzing the students' success achieved in two fifteen-minute tests (before and after the systematization classes) where they had to mark the correct statements (precisely formulated algebraic structures and homomorphisms). The results obtained by statistical analysis indicate that the students achieved statistically significantly better results in the test held after the systematization classes. In other words, the creation of mind maps by students gifted in mathematics had a positive effect on systematization of knowledge about Algebraic structures and on students' achievement in mathematics (specifically Linear Algebra and Analytical Geometry). This result implies that teachers who work with students gifted in mathematics should seriously consider organizing mathematics classes where students will systematize and deepen their theoretical knowledge by creating mind maps.

Keywords: mind maps, students gifted in mathematics, algebraic structures, teaching mathematics.

INTRODUCTION

Many studies and empirical research support the positive impact of using mind maps on learning focused on establishing connections between different and related concepts (Budd 2004; Farrand, Hussain, Hennessey 2002). Thus, positive results can be found in the literature of the use of mind maps created by teachers or students during the adoption of teaching content in mathematics (Brinkmann

aca.milenkovic.aca@gmail.com

2003). At the same time, it should be emphasized that the age of the students varies in different researches, from the youngest students of school age to high school students and participants in higher education (Farrand, Hussain, Hennessy 2002; Kovačević, Segedinac 2007). However, in most research, mind maps are used in heterogeneous classes of students, when it comes to students' achievements in mathematics. On the other hand, there is also a large amount of research related to gifted students in mathematics. The emphasis in those researches is mainly on: how to recognize these students, in particular at a younger age (Bicknell 2009); on mathematical giftedness and mathematical creativity (Parish 2014); on the development of a mathematics curriculum for students gifted in mathematics (Zmood 2014); in the choice and method of solving mathematical problems (Leikin 2009), etc. Therefore, although mind maps, as well as work with gifted students, represent the topics of a significant number of empirical researches, the amount of research that connects these two topics is practically negligible. Indeed, it is very difficult to find research that discusses mind mapping by gifted students in mathematics and the impact of mind mapping by gifted students in mathematics on their achievement in mathematics.

The aim of this research is reflected in the analysis of the impact of the methodological approach, which involves creating mind maps on behalf of students gifted in mathematics in order to deepen and systematize their theoretical knowledge, which is necessary for successfully solving concrete problems on the teaching topic of Algebraic structures.

THEORETICAL BACKGROUND

MIND MAPS

Throughout history, mainly due to the low technical-technological level of development, people used two-dimensional representations of their ideas to try to find a solution to a concrete problem, to perform classifications according to different criteria of various phenomena, or simply to present their ideas in a hierarchical order according to some principles. For these purposes, people used graphic representations of knowledge – mind maps, from the earliest times to the present day (Rhodes 2013). Mind maps are, formally, special diagrams that can be used in situations involving the need for learning and thinking in any form (Kovačević, Segedinac 2007). Using mind maps improves our intellectual potential: memory, thinking, and understanding and noticing relationships and connections between terms and concepts (Farrand, Hussain, Hennessy 2002; Papić, Aleksić, Kuzmanović, Papić 2015). As Buzan points out, the mind map as a powerful graphic tool can be the key to releasing the potential of the brain, and it has four basic characteristics, (Buzan, Buzan 1995):

- The subject of attention is crystallized in a central image.
- The main themes of the subject radiate from the central image as branches.
- Branches comprise a key image or keyword printed on an associated line. Topics of lesser importance are also represented as branches attached to higher level branches.
- The branches form a connected nodal structure.

Alamsyah (Alamsyah 2009) explains that mind map should have the following elements:

1) The center of the mind map is the main idea or idea.
2) The main branch or basic order ideas (BOI), the first level branch that radiates directly from the center of the map. Thoughts.
3) Branches, which are emanations from the main branch, can be written in all directions.
4) Words, using only keywords.
5) Pictures, using pictures they like.
6) Colors, using attractive colors on the map.

For creating a mind map, it is important to use the keywords. Keywords are words that represent the "trigger impulse" for more relevant associative meanings. Using keywords reduces the number of words that are used on the map but on the other hand does not reduce the quantity of information associated with those keywords. Sometimes, it can be difficult to find the keyword, if it is trapped in a sentence. On the other hand, another important aspect is that when we choose it, our mind "digs" deeper in search for new meanings. Using different colors is very useful and stimulating when creating mind maps (Kovačević, Segedinac 2007).

Of course, over the long period of development of human civilization, mind maps have evolved and are used in various segments of modern life. According to the authors of the book *Mind Maps* (Buzan, Buzan 1995; Buzan 1976), the number of people who began to use brilliant thinking and mind mapping grew by an almost logarithmic progression. Throughout history, examples of numerous creative people and thinkers who have used mind maps can be found. Some famous intellectuals and people who primarily used graphic-visual representations in their intellectual work were Leonardo da Vinci, then Albert Einstein who used mind maps in unconventional ways to create unconventional ways of thinking (Rhodes 2013). The famous and previously cited Tony Buzan, a British author, believes that literate and well-educated individuals are limited because they are unable to use many of the conceptual tools for thinking, including mind maps (Rhodes 2013).

According to the studies of cognitive psychology (Morita, Asada, Naito 2016), human understanding of knowledge is a complex and changeable imaging process. Psychology says that the human brain remembers images much more strongly than words. Humans have left and right brain hemispheres, which are responsible for different brain activities. The left hemisphere is responsible for words, logic, numbers, order, linearity, analysis, and lists, while the right is responsible for rhythm, imagination, colors, daydreaming, gestalt, and dimensions (Stanković, Ranđić 2008). Common methods of memorizing information force the individual's brain to work linearly and interfere with the natural functioning of the brain. The brain works by principle of association and based on that can connect an idea or data with many other ideas and concepts (Anokhin 1973). Conventional teaching methods better support the work of the left half of the brain compared with the right half of the brain, but using mind maps stimulates the work of both halves of the brain. In this way, the logical structures relate to imagination on paper, which is the basis for a mind map. The left side of the brain is activated by keywords on the map, while adding images, colors, and three-dimensionality activates the right brain hemisphere – "the right creative brain" (Svantesson 1992).

In today's insistence on quality education, more attention is given to the cultivation and promotion of students' active learning ability and their thinking ability. Since school-based learning is comprised from a set of multiple situations that involve solving problems, organizing data, taking notes, writing, and presentations, mind maps are offered as a tool for all these activities (Brinkmann 2003). Mind maps are considered an excellent tool for accelerating learning, creativity, solving complex problems, and saving time. Just designing mind maps imitates the work of the brain symbolically and visually on paper. They represent connections between concepts, which contributes to building better connections in the brain itself and better recall of information. Mind maps, therefore, reflect the natural functioning of the brain, because they have a branched radial structure branching from a central term (Buzan, Buzan 1999). According to some researches, mind maps have a positive influence on the understanding of abstract concepts (Roth, Roychoudhury 1992). The observations that the individual creating the map play an important role in the placement, assimilation, organization, and retention of data (Ornstein 1986; Ornstein 1991). Mind mapping promotes divergent and creative thinking (White, Gunstone 1992). Connections between different parts of the map can be obtained by linking different parts of the map with arrows. This makes it easy to examine patterns of thought and similarities and connections between information in different parts of the map.

Mind maps can be an essential tool for teaching and learning. To carry out the steps of constructing a mind map, we must first understand the content of knowledge, proceed to identify the core content, and divide it into main ideas and identify sub-ideas of each main idea (Buzan, Buzan 1999). Many studies point to the effectiveness of the mind map technique (Budd 2004; Farrand, Hussain, Hennessey 2002). Efficiency of use of mind map techniques when improving factual knowledge from written information was studied by Farrand, Hussain, and Hennessey (Farrand, Hussain, Hennessey 2002). The attention of researchers was focused on mind maps as a learning aid. The remembered content was stable in both groups, but the participants from the group that used the mind mapping technique remembered the content better after a week. The authors pointed out that this method has an advantage over conventional methods of learning, and that students were enthusiastic about this method, which lead to more effective training for the implementation of the curriculum. In the research of Budd (Budd 2004), mind maps are presented as a tool for overcoming traditional blackboard and chalk learning styles. The work shows the possibility of using mind maps for the purpose of different learning styles and renewing energy during the semester. The exercise was organized so that students create within one subject mind maps on the given teaching topic. In groups of three, students were asked to think about what the first step in the formation of mind maps is. During the exercise, the instructor moved among the students and gave them feedback on the process of creating the map. This research supported the idea of active learning, and students with higher scores agreed on the positive impact of learning based on mind maps. Nowadays there are numerous software tools that enable the creation of mind maps, such as: Coggle, Freemind, Xmind, MindMeister, MindManager, LucidChart, Microsoft Visio, ClickUp, etc.

It is also possible to use mind maps in mathematics education. According to Brinkmann (Brinkmann 2003), mind maps can help in organizing information, they can be used as an aid in memorizing content and its repetition, and then in connecting new information with the students' existing knowledge. They allow students' cognitive structures to become visible, promote creativity, and ultimately help students to see the connection between mathematics and the real world. Although, as the author points out, mind maps are rarely used in mathematics education, feedback indicates that students who were not good at mathematics benefited from mind mapping. They understood the relationships and connections between mathematical concepts while creating a mind map (Brinkmann 2003). Mind maps made a strong impression on students who usually memorize. They turned such habits into meaningful learning (Arifah, Suyitno, Rachmani Dawi, Kelud Udara 2020).

STUDENTS GIFTED IN MATHEMATICS

The concept of giftedness is popularly considered as a concept that articulates the highest level of intelligence determined by IQ tests. Giftedness is a much wider concept, which refers to an alignment which is both cognitive and emotional and includes unique developmental aspects, as well as familial and social aspects (Tamir 2012, as cited in Zedan, Bitar 2017).

Educational literature related to the issues of high mathematical ability, mathematical talents, mathematical giftedness, and mathematical creativity contain a variety of descriptive reports and instructional guidelines, but there are much fewer research reports that could be found on the issues related to mathematical talents and mathematical giftedness. Schoenfeld (Schoenfeld 2000; Schoenfeld 2002) expressed the two main purposes of research in mathematics education which could be maintained for the research in the field of mathematical giftedness and creativity. Those purposes are:

- First (theoretical) is to understand the nature of mathematical giftedness and mathematical creativity from the perspectives of thinking, teaching, and learning;
- Second (applied) is to use such understanding in improving mathematics instruction in a way that helps realize mathematical giftedness and encourage mathematical creativity.

According to Leikin and her colleagues (Leikin 2009; Leikin 2014; Leikin, Paz-Baruch, Waisman, Lev 2017), the domain of mathematical giftedness implies a collection of certain mathematical abilities and personal qualities. Students who are gifted in mathematics are described as students with strong problem-solving abilities, metacognitive abilities, creative mathematical thinking, and high ability/performance in mathematical problem-solving. Characteristics of students that can indicate mathematical giftedness usually include: an extraordinary curiosity for numbers and mathematical information, a capability to understand and implement mathematical concepts quickly, a distinctively high ability to recognize patterns and abstract thinking, flexibility and creativity in strategies for problem solution, an ability to move mathematical concept to an unfamiliar situation, as well as perseverance in solving challenging problems (Stepanek 1999). Mathematically gifted individuals possess intellectual characteristics, such as curiosity, the ability to visualize models, fast thinking, and metaphorical thinking (Deary 2000; Silverman 1997). Krutetskii (Krutetskii 1976: 77) implies that gifted and talented mathematics students have (among other capacities) “the ability for rapid and broad generalization of mathematical relations and operations, and flexibility of mental processes”. The teachers observed the different pace of mathematics learning, an intuitive mathematical knowledge in problem-solving, their interest in mathematics, the sense of humor and ability to think in more abstract terms than peers of the same age, as well as more mental flexibility and a discourse based on logical thinking characterized students gifted in mathematics (Bicknell 2008). According to the students, other aspects that confirmed their mathematical giftedness include success in competitions, competence with basic mathematical facts, speed of computational skills, problem-solving abilities, and capacity to work on “special projects” or on more/different work (than their classmates) to complete independently (Bicknell 2008; Subotnik, Robinson, Callahan, Gubbins 2012).

Sriraman (Sriraman 2009) claims that mathematical creativity could be considered as the main mechanism of the growth of mathematics as science. Mathematical creativity is also mentioned as a characteristic among students gifted in mathematics, even though there is no commonly accepted definition of that term (Plucker, Beghetto, Dow 2004; Singer, Sheffield, Leikin 2017). Other studies take a different approach to creativity and adopt the concept of cognitive flexibility, which is explained as an interlude between cognitive variety, cognitive novelty, and changes in cognitive framing (Schoevers, Kroesbergen, Kattou 2020; Zhang, Gan, Wang 2017). Mathematical creativity also promotes self-efficacy (Bicer, Lee, Perihan, Capraro, Capraro 2020; Regier, Savic 2020).

Bicknell and Holton (Bicknell, Holton 2009) argued that mathematical giftedness can be manifested in three ways. The first is the analytic mode – mathematically gifted students figure out problems by using logic and thought. The second is the geometric mode – students will prefer to use sketches and visual aids to figure out problems. The third is the harmonic mode, which represents the gifted students who are capable of both the analytic and the geometric modes.

When it comes to the mathematics teacher who teaches students gifted in mathematics, they should "have access to professional development research information and resources to deal with such issues as identification or recognition of students with mathematical promise, high levels of expectations for all students along with challenging top students to even higher levels of success, pedagogical and questioning techniques to extend students' thinking, and selection and/or development of appropriate curriculum and assessment tools that provide opportunities for students to create problems, generalize patterns, and connect various aspects of mathematics, development of teachers' own mathematical power to make connections and the mathematical sophistication to see the big picture, making appropriate instructional decisions for these promising students, and awareness of, access to and ability to use technology and other tools" (Singer, Sheffield, Leikin 2017: 29). In addition, teachers should continue to strengthen their own mathematical content knowledge and demonstrate the joy of being a lifelong learner of mathematics (Sheffield, Bennett, Berriozabal, DeArmond, Wertheimer 1999). Hoth (2017) suggests that the main element in fostering mathematically gifted students is giving them different learning opportunities. One way to do that could be by creating mind maps during their mathematics classes by students gifted in mathematics.

CHARACTERISTICS OF THE TOPIC OF ALGEBRAIC STRUCTURES

Here are the few definitions for the algebraic structures, from simpler to more complex structures, that students should adopt in the third year of high-school education in the program of the subject Linear Algebra and Analytic Geometry, for the students gifted in mathematics:

Ordered pair $(G,*)$*, where* G *is nonempty set closed under binary operation* $*$ *is groupoid.*

A semigroup $(G,*)$ *is groupoid where the binary operation* $*$ *is associative.*

A monoid is a semigroup with an identity (neutral) element.

A group is a monoid such that each $a \in G$ *has an inverse* $a^{-1} \in G$.

Group G *is Abelian or commutative if* $a*b=b*a$ *for all* $a,b \in G$ *(if binary operation* $*$ *is commutative).*

After Abelian groups, students should learn the algebraic structures with two binary operations: ring, ring with neutral element, and field. Also, students should adopt structure – preserving maps: homomorphism (mapping between two groupoids $(G,*)$ and $(H,\cdot)$ where $f:G \to H, (\forall x,y \in G)\ f(x*y)=f(x)\cdot f(y)$), endomorphism (homomorphism which maps G to G), monomorphism (homomorphism which is also injection), epimorphism (homomorphism which is also surjection), isomorphism (homomorphism which is also injection and surjection), and automorphism (homomorphism which is also injection, surjection and which maps G to G),

There is one common feature about the way that algebraic structures and structure-preserving maps are defined. That feature reflects that more complex structures and mappings are defined through introducing the new property to an already defined mathematical concept (algebraic structure or structure-preserving map). These definitions could be considered as analytic definitions. Under this type of definition, we consider the definitions of the nearest genus concepts and their differences. Aristotle described them as: *Genus proximum et differentia specifica.* For instance, in the definition: *A monoid is a semigroup with an identity element*, semigroup is the nearest genus concept to monoid, and existence of the identity element differentiates the concept of a monoid from a semigroup. With many analytic definitions, as the number of genus concepts and differences is increasing, it gets more and more complex for the students to memorize and adopt all these concepts and connect them in a proper mental scheme.

The importance of having sufficient theoretical knowledge regarding the theme of algebraic structures is high because in most of the concrete problems in this theme, students need to examine the type of given algebraic structure with one binary operation, algebraic structure with two binary operations, and the type of structure-preserving maps. For solving these kinds of problems, students must understand what properties they should examine to determine the type of the given algebraic structure (or mapping), and it is quite hard to memorize properties for all these mathematical concepts individually. So, the best approach for learning these mathematical concepts is to know the relationships between these concepts. This puts the teacher in a position in which he must design and conduct well-structured systematization mathematics classes.

RESEARCH METHODOLOGY

As stated earlier, for the students to adopt the appropriate algebraic structures with understanding, to understand the relations between them as much as possible, and to later apply the theoretical knowledge to concrete problems, it is important that the students systematize the necessary theoretical knowledge. Having in mind that the concepts of algebraic structures are quite abstract and that students don't have previous experience with these concepts, it is important that teacher organize well-structured systematization classes.

For this purpose, it is planned that students systematize appropriate theoretical knowledge by creating mind maps, since mind maps have many of the aforementioned positive aspects. The goal of this research is to determine the effects of creating mind maps by students gifted in mathematics in order to improve their achievements, i.e. to improve their necessary theoretical knowledge.

PROCEDURE AND INSTRUMENTS

In the class which preceded the experimental class, the students were asked to bring thicker and larger paper, as well as crayons and pens in various colors. When asked by the teacher if they had created mind maps during their education so far, 7 or 8 students stated that they had in different subjects during their education in elementary school (geography, biology, and Serbian language), and 3 students made mind maps in math classes in elementary school, but during their high school education they did not create a mind map in any mathematics (mathematical analysis, algebra, or geometry) class.

In the introductory part of the systematization classes, the teacher told the students that their task in the given classes was to create a mind map on which they were to show algebraic structures with one and two binary operations, as well as to create a mind map on which they were to illustrate and connect contents related to mappings. Then the teacher explained to them how the given contents should be connected, with the suggestion that they first prepare a working version of the mind map on a smaller piece of paper, and then, when they create a picture in their minds of how it should look according to their understanding, to translate it into the larger, final form of the mind map. In the first part of the classes, students had the task of presenting the algebraic structures with one and with two binary operations, and in second part of the class, students had the task of presenting mappings (homomorphisms) together with special cases of homomorphisms (endomorphism, epimorphism, monomorphism, isomorphism, automorphism) through a mind map.

During the process of creating mind maps, students flipped through notebooks and textbooks, thus determining the acquired knowledge and then systematically presenting them on paper, in the form of mind maps. It should be emphasized that students created mind maps by working in pairs. The students chose who they would collaborate with, with the aim that during the work the students openly talk, discuss, exchange their opinions, and choose the best ways to present appropriate teaching and learning content. In this way, in addition to developing specific subject competencies, students also developed cross-curricular competencies: lifelong learning, communication, and cooperation.

In order to examine the effects of the given methodological approach, students were tested before and after the systematization classes. Namely, in the final part of the class, which preceded the class of systematization, the students solved Test 1 (see Appendix), in which some correct statements were formulated, as well as statements that were essentially incorrect but were formulated similarly to the correct statements. In the given statements, the students were required to show that they recalled and understood the relations between different but related algebraic structures such as groupoid and semigroup, semigroup and monoid, monoid and group, and then mappings such as homomorphism and monomorphism, monomorphism and isomorphism, epimorphism and automorphism, etc. Therefore, the students were not required to simply reproduce the formulations of the definitions of algebraic structures by stating all the conditions that must be satisfied by the given operation(s) defined on the given set, i.e., all properties of the mapping, but to recognize (based on their knowledge of their properties) which structures are special cases of other structures, i.e., under which new conditions a given structure becomes a structure that represent another, higher level class of structures. Students completed the test by marking the correct statements (by circling the letter in front of the correct statements), while the incorrect statements were to remain unmarked. This was followed by a systematization class where students created mind maps and thus connected their knowledge in suitable schemes. Immediately at the beginning of the class that followed the systematization class, the students were given Test 2, designed in accordance with Test 1, where again some statements were correct, and some were not.

PARTICIPANTS

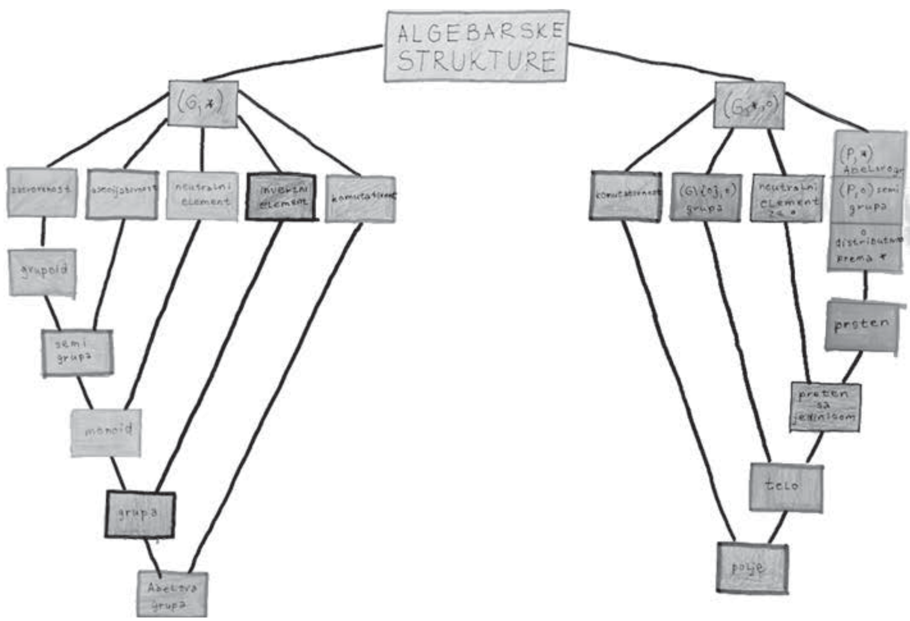
Participants in this quasi-experimental study were 17 third year students gifted in mathematics from the First Gymnasium in Kragujevac who participated in the subject Linear Algebra and Analytic Geometry in the 2021/2022 school year.

RESULTS AND DISCUSSION

ANALYSIS OF THE STUDENTS' WORK IN THE CLASSES

From the students' work, four mind maps are chosen to represent the quality and the mutual characteristics of the mind maps created by students.

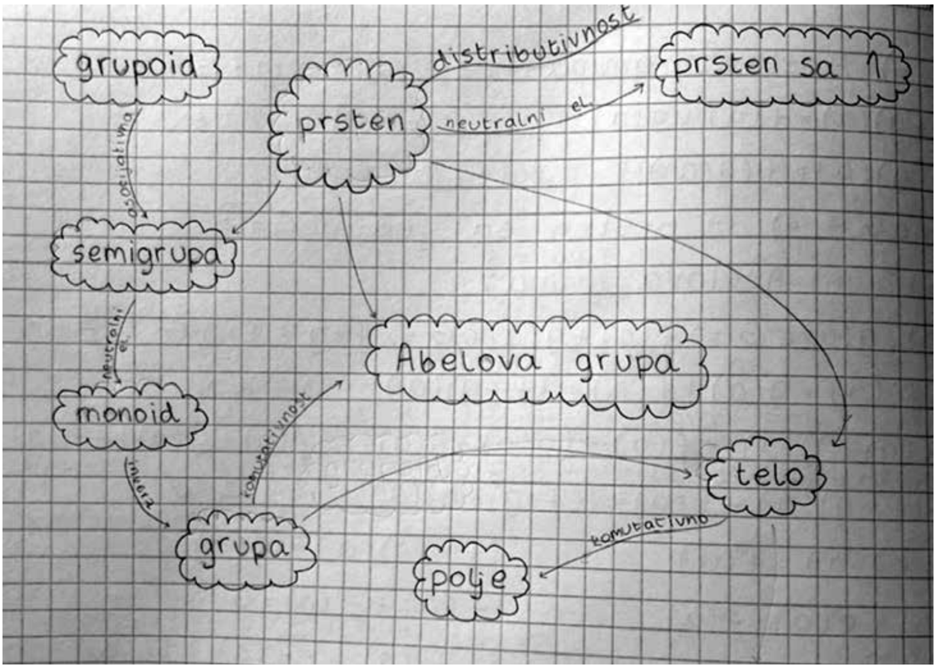
Figure 1. Pair 1 mind map for the theme Algebraic structures



The image presented in the Figure 1 shows a mind map on which the students in the central part, in accordance with the instructions, presented a key term (algebraic structures), then divided the given mathematical concepts into two parts: structures with one binary operation on the left side of the mind map and structures with two binary operations on the right side of the mind map. The students also chose to present the concepts in order of the complexity of the algebraic structures, that is, the number of conditions that the structure must fulfill (the complexity of the structures increases when moving from top to bottom).

The following image presented in the Figure 2 shows a mind map that does not follow the pre-agreed upon structure which a mind map should have. Namely, we can see that to a certain extent the concepts are linearly represented (from groupoid to group), so that the Abelian group is represented in the central part of the mind map, while structures with two binary operations are shown both above (ring, ring with neutral element) and below the centrally represented term (field).

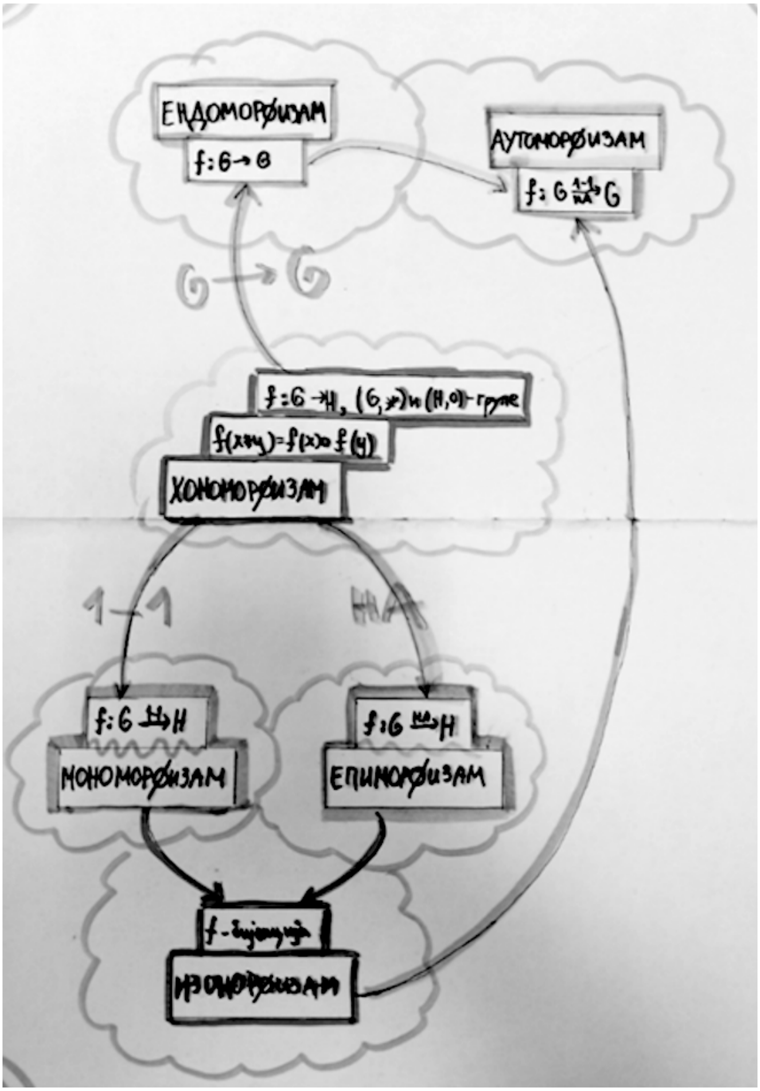
Figure 2. Pair 2 mind map for the theme Algebraic structures



In contrast to the first mind map (shown in Figure 1), the second mind map (shown in Figure 2) is presented more confusedly. It does not have an ideal structure, but still, from the perspective in which the mathematical concepts are connected, it can be concluded that the students master the given concepts and understand their properties and the connections between them.

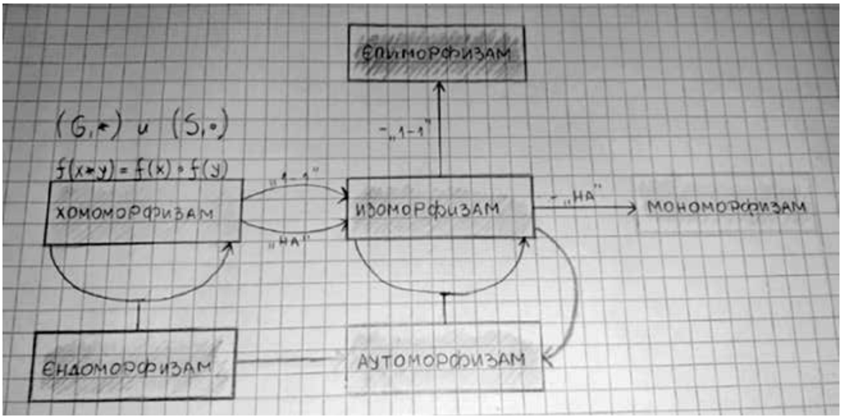
In the Figure 3, we can see a very nicely structured mind map, on which the basic concept (homomorphism) is presented in the central part, together with the definition written in mathematical notation. Furthermore, it can be observed that concepts further branch according to special classes of homomorphisms with precisely written conditions in mathematical notation as well.

Figure 3. Pair 6 mind map for the theme Homomorphisms



The image presented in Figure 4 shows a not quite satisfactorily structured mind map. Namely, the concept of homomorphism is not presented in the central part, but the concept of isomorphism (which represents a homomorphism that is also a bijection). This caused the terms epimorphism (which represents a homomorphism that is a surjection, not an injection) and monomorphism (which represents a homomorphism that is an injection, not a surjection) to be presented as an isomorphism in which one of the properties does not apply (with a "minus" sign).

Figure 4. Pair 3 mind map for the theme Homomorphisms



The first impression based on the analysis of students' work on creating mind maps in the systematization classes is that some students (probably due to a lack of experience in systematically presenting teaching and learning content through mind maps) did not follow the technical instructions for making mind maps, specifically about the way in which concepts should be arranged. The use of colors is also not quite satisfactory. On the other hand (which is extremely important, and which speaks in favor of the fact that the students showed an enviable level of knowledge), there were no material errors in students' work. There were no errors of a mathematical nature on any mind map. In all mind maps, the conditions that certain mathematical concepts must meet were accurately and precisely represented.

Students who took part in the quasi-experimental study were very enthusiastic while creating mind maps and very dedicated to their work. These students' impressions are in accordance with the conclusion of other research (Budd 2004) about students' recognition of the positive impact of learning based on mind maps.

ANALYSIS OF THE STUDENTS' TESTS RESULTS

As said earlier, both tests consisted of a total of 16 statements (see Appendices).

Table 1. Students' results on the Test 1 and the Test 2

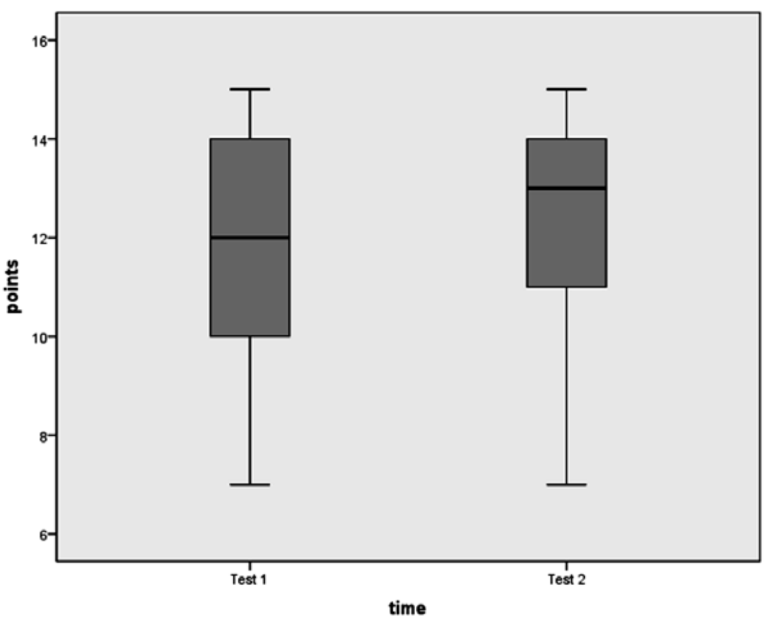
Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Before	14	14	7	12	10	12	7	7	12	10	9	15	14	12	14	11	11
After	15	14	11	11	11	14	7	13	14	11	12	13	14	14	14	12	13
Difference	1	0	4	-1	1	2	0	6	2	1	3	-2	0	2	0	1	2

The results of the students' work on these tests are presented in Table 1. As can be seen, for each student who attended all four classes (systematization classes, and classes before and after the systematization classes) the differences (number of points that students achieved after creating mind map minus the number of points students achieved before creating mind maps) in the number of points scored by the students were calculated. Out of a total of 17 students, 12 students achieved a higher number of points on Test 2 compared to the number of points on Test 1. Of the remaining 5 students, 3 students achieved identical results, while two students had more incorrect answers after the systematization classes.

It is interesting that the students who showed the greatest progress in their results (i.e., their knowledge of the given concepts) are those students who achieved lower results and showed a lower degree of interest for the given teaching contents in the third grade within this subject.

Based on the graphic presented in Figure 5, it can be seen from the distribution of the number of points (that students achieved in Test 1 and Test 2), that in most cases, students achieved between 10 and 14 points on Test 1 and between 11 and 14 points on Test 2. Minimums and maximums of points that students achieved are the same for both the tests. Also, the median number of points that students achieved on Test 2 is higher than on Test 1.

Figure 5. Distribution of the number of points that students achieved while solving Test 1 and Test 2



Arithmetic means of the number of points scored by students before (Test 1) and after the systematization classes (Test 2) were calculated. The average number of points achieved by the students before the systematization classes is equal to 11.24, while the average number of points achieved by the students after the systematization classes is equal to 12.54. So, on average, students improved their scores by 1.3 points. Bearing in mind that they could achieve a maximum of 16 points, we notice that the students generally showed an enviable level of knowledge both before and after the systematization classes. This speaks in favor of the fact that during the classes of adopting new teaching and learning content and exercise classes as well, the students adopted and understood the teaching content to a large extent, while after the classes of systematization, which was conducted through the creation of mind maps by the students, they additionally established appropriate connections and relations between different mathematical concepts. This result is in line with other results regarding the potential of using mind maps in order for students to deepen their knowledge (Brinkmann 2003; Kovačević, Segedinac 2007; Papić, Aleksić, Kuzmanović, Papić 2015).

Table 2. Statistical analysis of the students' results

Time	Number of students	Means	Medians	Mean rank	Sum of ranks	Wilcoxon Signed Ranks Test	
						Z	p (2-tailed)
Before	17	11.24	12.00	5.50	11.00	- 2.441	0.015
After	17	12.54	13.00	7.27	80.00		

Based on the results of the non-parametric Wilcoxon rank test, it was found that the number of points that students achieved on Test 2, i.e., on the test which followed the systematization classes, were statistically significantly better compared to the number of points the students achieved on the test held before the systematization classes (Test 1). As earlier confirmed in other empirical research conducted on heterogeneous classes of students (Kovačević, Segedinac 2007), this result speaks in favor of the fact that the students gifted in mathematics significantly systematized and deepened their theoretical knowledge (about algebraic structures and homomorphism) by creating mind maps and systematizing the teaching and learning content, i.e. mind maps contribute to better student achievement when their creation (by students) is implemented in homogeneous classes of students (classes formed with students gifted in mathematics).

CONCLUSION

It is generally known that the teaching contents provided by the Serbian curriculum for secondary school (high school) students are significantly more abstract compared to elementary school, while the teaching methodology is also significantly more formalized. Mathematics teachers, withdrawn from the teaching content, are mostly implementing the frontal form of teaching mathematics, which is even more pronounced in classes with students gifted in mathematics. Examples of some more innovative approaches (not related to solving tasks), except perhaps the occasional use of ICT in teaching with gifted students for mathematics, are very difficult to find, at least in the relevant literature. On the other hand, mind maps have proven to be effective in the implementation of mathematics classes with the aim of students acquiring and understanding the necessary knowledge and connecting mathematical concepts in an appropriate scheme. All these reasons can be considered as the background for the highly motivated students who participated in this quasi-experimental study that aimed to examine whether the creation of mind maps by gifted students in mathematics leads to better student achievement. Indeed, all students participated in the work during the classes and showed an enviable level of commitment.

Based on the analysis of the students' work, it can be concluded that a certain number of students, probably due to the lack of experience in the creation of mind maps, bypassed some agreed technical characteristics that a mind map should fulfill. On the other hand, all mind maps were mathematically correct, with appropriate mathematical notation and without material errors. The results of the tests that the students took before and after the systematization classes indicate that the creation of a mind map with the aim of systematically presenting the teaching content from the Algebraic Structure topic leads to an improvement in student mathematical achievement.

Bearing in mind the small sample size of this research, as well as the fact that the students only systematized the teaching content from one topic in this way, no generalized conclusions can be made, but the results of this quasi-experimental study certainly speak in favor of the implementation of systematization classes in mathematics couses (in analysis, algebra, geometry courses) with students gifted in mathematics. Some future research could follow in order to design and implement several systematization classes during one school year with the students gifted in mathematics (on one or even on two mathematical courses with the same group of students) and additionally to examine the effects of this methodological approach.

REFERENCES

Alamsyah (2009): M. Alamsyah, *Tips to Improve Achievement With Mind Mapping* (translated from Indonesian), Jakarta, Indonesia.

Arifah, Suyitno, Rachmani Dewi, Kelud Utara (2020): U. Arifah, H. Suyitno, N. Rachmani Dewi, J. Kelud Utara, Mathematics Critical Thinking Skills based on Learning Styles and Genders on Brain-Based Learning Assisted by Mind-Mapping, *Unnes Journal of Mathematics Education Research*, 11(1).

Anokhin (1973): P. K. Anokhin, The forming of natural and artificial intelligence, *Impact of Science in Society*, 23(3).

Bicer, Lee, Perihan, Capraro, Capraro (2020): A. Bicer, Y. Lee, C. Perihan, M. M. Capraro, R. M. Capraro, Considering mathematical creative self-efficacy with problem posing as a measure of mathematical creativity, *Educational Studies in Mathematics*, 105, 457–485. doi: 10.1007/s10649-020-09995-8.

Bicknell (2008): B. Bicknell, Who are the mathematically gifted? Student, parent, and teacher perspectives, *Proceedings of ICME11. TG6: Activities and Programs for Gifted Students.*

Bicknell, Holton (2009): B. Bicknell, D. Holton, Gifted and Talented Mathematics Students, In: R. Averill, R. Harvey (Eds.), *Teaching Secondary School Mathematics and Statistics: Evidence-Based Practice*, 1, 173–186, Wellington: NZCER Press.

Brinkmann (2003): A. Brinkmann, Mind Mapping as a Tool in Mathematics Education, *National Council of Teachers of Mathematics*, 96(2), 96–101.

Budd (2004): J. W. Budd, Mind Maps as Classroom Exercises, *The Journal of Economic Education*, 35(1), 35–46.

Buzan, Buzan (1995): T. Buzan, B. Buzan, *The Mind Map Book*, London: BBC Books.

Buzan (1976): T. Buzan, *Use both sides of your brain*, Dutton, New York.

Deary, (2000): I. J. Deary, Simple information processing and intelligence, In: R. J. Sternberg (Ed.), *Handbook of Intelligence*, Cambridge: Cambridge University Press, 267–284.

Farrand, Hussain, Hennessy (2002): P. Farrand, F. Hussain, E. Hennessy, The efficacy of the 'mind map' study technique, *Medical education*, 36, 426–431.

Hoth, Kaiser, Busse, Döhrmann, König, Blömeke (2017): J. Hoth, G. Kaiser, A. Busse, M. Döhrmann, J. König, S. Blömeke, Professional competences of teachers for fostering creativity and supporting high-achieving students, *ZDM International Journal of Mathematics Education*, 49, 107–120. doi: 10.1007/s11858-016-0817-5.

Jensen (1973): L. R. Jensen, *The relationships among mathematical creativity, numerical aptitude, and mathematical achievement*, unpublished dissertation, Austin, TX: The Univ. of Texas at Austin.

Krutetskii (1976): V. A. Krutetskii, *The psychology of mathematical abilities in schoolchildren* (translation from Russian), Chicago: The University of Chicago Press.

Kovačević, Segedinac (2007): J. Ковачевић, М. Сегединац, Допринос реформи наставе – мапе ума, *Зборник Матице српске за друштвене науке*, 122, 191–201.

Leikin (2009): R. Leikin, Exploring mathematical creativity using multiple solution tasks, In: R. Leikin, A. Berman, B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*, 129–145, Rotterdam: Sense Publishers.

Leikin (2014): R. Leikin, Giftedness and high ability in mathematics, In: S. Lerman (Ed.), *Encyclopedia of Mathematics Education*, Dordrecht: Springer, 247–251.

Leikin, Leikin, Paz-Baruch, Waisman, Lev (2017): R. Leikin, M. Leikin, N. Paz-Baruch, I. Waisman, M. Lev, On the four types of characteristics of super mathematically gifted students, *High Ability Studies*, 28, 107–125. doi: 10.1080/13598139.2017.1305330.

Morita, Asada, Naito (2016): T, Morita, M. Asada, E, Naito, Contribution of Neuroimaging Studies to Understanding Development of Human Cognitive Brain Functions, *Frontiers in Human Neuroscience*, 10, 1–14 doi: 10.3389/fnhum.2016.00464.

Ornstein (1986): R. Ornstein, *Multimind: A new way of looking at human behaviour*, Boston: Houghton Mifflin.

Ornstein (1991): R. Ornstein, *The evolution of consciousness*, New York: Prentice Hall Press.

Papić, Aleksić, Kuzmanović, Papić (2015): Ž. M. Papić, V. Aleksić, B. Kuzmanović, M. Ž. Papić, Primena mape uma i konceptualnih mapa u nastavnom procesu, *Vaspitanje i obrazovanje*, 12(3), 13–25.

Parish (2014): L. Parish, Defining mathematical giftedness, In: J. Anderson, M. Cavanagh, A. Prescott (Eds.), Curriculum in focus: Research guided practice, *Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia*, 509–516, Sydney: MERGA.

Plucker, Beghetto, Dow (2004): J. A. Plucker, R. A. Beghetto, G. T. Dow, Why isn't creativity more important to educational psychologists? Potentials, pitfalls, and future directions in creativity research, *Educational Psychology*, 39, 83–96. doi: 10.1207/s15326985ep3902_1

Regier, Savic (2020): P. Regier, M. Savic, How teaching to foster mathematical creativity may impact student self-efficacy for proving, *Journal of Mathematical Behavior*, 57(2). doi: 10.1016/j.jmathb.2019.100720

Rhodes (2013): S. Rhodes, *Mind Maps!*, JJ Fast Publishing, LLC.

Roth, Roychoudhury (1992): W. Roth, A. Roychoudhury, The Social Construction of Scientific Concepts or the Concept Map as Device and Tool Thinking in High Conscription for Social School Science, *Science Education*, 76(5), 531–557.

Schoenfeld (2000): A. H. Schoenfeld, Purposes and methods of research in mathematics education, *Notices of the American Mathematical Society*, 47, 2–10.

Schoenfeld (2002): A. H. Schoenfeld, Research methods in (Mathematics) education, In: L. English (Ed.), *Handbook of international research in mathematics education*, 435–488, Mahwah, NJ: Erlbaum.

Schoevers, Kroesbergen, Kattou (2020): E. M. Schoevers, E. H. Kroesbergen, M. Kattou, Mathematical creativity: a combination of domain-general creative and domain-specific mathematical skills, *Journal of Creative Behavior*, 54, 242–252. doi: 10.1002/jocb.361.

Sheffield, Bennett, Berriozabal, DeArmond, Wertheimer (1999): L. J. Sheffield, J. Bennett, M. Berriozabal, M. DeArmond, R. Wertheimer, Report of the NCTM task force on the mathematically promising, In: L. J. Sheffield (Ed.), *Developing mathematically promising students*, 309–316, Reston, VA: NCTM.

Silverman (1997): L. K. Silverman, The construct of asynchronous development, *Peabody Journal of Education*, 72, 36–58.

Singer, Sheffield, Leikin (2017): F. M. Singer, L. J. Sheffield, R. Leikin, Advancements in research on creativity and giftedness in mathematics education: introduction to the special issue, *ZDM International Journal of Mathematics Education*, 49, 5–12. doi: 10.1007/s11858-017-0836-x.

Sriraman (2009): B. Sriraman, The characteristics of mathematical creativity, *ZDM International Journal of Mathematics Education*, 41, 13–27.

Stanković, Ranđić (2008): N. Stanković, S. Ranđić, Primena mentalnih mapa u nastavi, u: D. Golubović (ur.), *Zbornik radova naučno-stručnog skupa Tehnika i informatika u obrazovanju – TIO'08*, 214–220, Čačak: Tehnički fakultet Čačak.

Stepanek (1999): J. Stepanek, *Meeting the Needs of Gifted Students: Differentiating Mathematics and Science Instruction*, United States of America: Northwest Regional Educational Laboratory.

Subotnik, Robinson, Callahan, Gubbins (2012): R. F. Subotnik, A. Robinson, C. M. Callahan, E. J. Gubbins, *Malleable minds: Translating insights from psychology and neuroscience to gifted education*, Storrs: Univ. of Connecticut, NRCGT.

Svantesson (1992): I. Svantesson, *Mind Mapping und Gedächtnistraining*, Bremen: GABAL.

White, Gunstone (1992): R. T. White, R. Gunstone, *Probing Understanding*, New York: Falmer Press.

Zedan, Bitar (2017): R. Zedan, J. Bitar, Mathematically gifted students: their characteristics and unique needs, *European Journal of Education Studies*, 3(4), 236–260.

Zhang, Gan, Wang (2017): L. Zhang, J. Q. Gan, H. Wang, Neurocognitive mechanisms of mathematical giftedness: a literature review, *Applied Neuropsychology: Child*, 6, 79–94, doi: 10.1080/21622965.2015.1119692.

Zmood (2014): S. Zmood, Fostering the promise of high achieving mathematics students through curriculum differentiation, In: J. Anderson, M. Cavanagh, A. Prescott (Eds.), *Curriculum in focus: Research guided practice* (Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia), 677–684, Sydney: MERGA.

APPENDICES

TEST 1

Mark the correct claims.

a) If algebraic structure $(G,*)$ is groupoid, then $(G,*)$ is semigroup.
b) If algebraic structure $(G,*)$ is semigroup, then $(G,*)$ is groupoid.
c) If algebraic structure $(G,*)$ is semigroup, then $(G,*)$ is monoid.
d) If algebraic structure $(G,*)$ is monoid, then $(G,*)$ semigroup.
e) If algebraic structure $(G,*)$ is monoid in which every element has its inverse element, then $(G,*)$ is Abelian group.
f) If algebraic structure $(G,*)$ is group and if the binary operation $*$ is associative on the set G, then $(G,*)$ is Abelian group.
g) If algebraic structure $(G,+,\cdot)$ is ring and if $(G\setminus\{0\})$ is group, then $(G,+,\cdot)$ is field.
h) If algebraic structure $(G,+,\cdot)$ is ring and if $(G\setminus\{0\})$ is Abelian group, then $(G,+,\cdot)$ is field.
i) If algebraic structure $(G,+)$ is Abelian group, if $(G\setminus\{0\})$ is Abelian group and if multiplication is distributive over addition, then $(G,+,\cdot)$ is field.

j) If algebraic structure $(G,+,\cdot)$ is field, then $(G,+,\cdot)$ is ring.
k) If algebraic structure $(G,+,\cdot)$ is ring, then $(G,+,\cdot)$ is field.
l) If structure-preserving map is monomorphism, then it is homomorphism.
m) If structure-preserving map is isomorphism, then it is monomorphism.
n) If structure-preserving map is endomorphism, then it is automorphism.
o) If structure-preserving map is isomorphism, then it is epimorphism.
p) If structure-preserving map is isomorphism, then it is automorphism.

TEST 2

Mark the correct claims.
a) If algebraic structure $(G,*)$ is groupoid, then $(G,*)$ is monoid.
b) If algebraic structure $(G,*)$ is monoid, then $(G,*)$ is groupoid.
c) If algebraic structure $(G,*)$ is semigroup, then $(G,*)$ is monoid.
d) If algebraic structure $(G,*)$ is monoid, then $(G,*)$ semigroup.
e) If algebraic structure $(G,*)$ is semigroup in which every element has its inverse element, then $(G,*)$ is group.
f) If algebraic structure $(G,*)$ is group and if the binary operation $*$ is commutative on the set G, then $(G,*)$ is Abelian group.
g) If algebraic structure $(G,+)$ is Abelian group, if $(G\setminus\{0\})$ is group and if multiplication is distributive over addition, then $(G,+,\cdot)$ is field.
h) If algebraic structure $(G,+,\cdot)$ is field, then $(G,+,\cdot)$ is ring.
i) If algebraic structure $(G,+,\cdot)$ is ring, then $(G,+,\cdot)$ is field.
j) If algebraic structure $(G,+,\cdot)$ is ring with neutral element, then $(G,+,\cdot)$ is field.
k) If algebraic structure $(G,+,\cdot)$ is field, then $(G,+,\cdot)$ is ring with neutral element.
l) If structure-preserving map is monomorphism, then it is epimorphism.
m) If structure-preserving map is isomorphism, then it is monomorphism.
n) If structure-preserving map is endomorphism, then it is epimorphism.
o) If structure-preserving map is isomorphism, then it is automorphism.
p) If structure-preserving map is automorphism, then it is monomorphism.

Александар З. Миленковић
Универзитет у Крагујевцу
Природно-математички факултет
Институт за математику и информатику
Крагујевац

АЛГЕБАРСКЕ СТРУКТУРЕ И ИЗРАДА МАПА УМА ОД СТРАНЕ УЧЕНИКА СА ПОСЕБНИМ СПОСОБНОСТИМА ЗА МАТЕМАТИКУ

Резиме: Рад са ученицима надареним за математику је предмет великог броја студија. Такође, у литератури се могу наћи примери позитивног утицаја употребе мапа ума на учење са разумевањем, повезивањем појмова у одговарајуће схеме, али утицај креирања мапа ума од стране ученика са посебним способностима за математику на њихова постигнућа није довољно истражен. Имајући то у виду, као и да алгебарске структуре представљају наставну тему у којој је неопходно да ученици усвоје одговарајућа теоријска знања о поменутим алгебарским структурама и односима између њих, примењен је овај методски приступ како би ученици дате појмове повезали у одговарајућу шему. У том циљу спроведена су два часа (двочас) систематизације за наставну тему Алгебарске структуре тако што су ученици креирали по две мапе ума (једну за алгебарске структуре са једном и са две бинарне операције и другу за хомоморфизме). Ефекти овог приступа на часовима систематизације су испитивани анализом успеха ученика који су они остварили приликом израде два петнаестоминутна теста (пре и после часова систематизације) на којима је требало да означе тачне тврдње (прецизно формулисане алгебарске структуре и хомоморфизме). Резултати добијени статистичком анализом указују да су ученици постигли статистички значајно боље резултате на тесту одржаном након часова систематизације (у односу на резултате постигнуте на тесту одржаном пре часова систематизације). Другим речима, креирање мапа ума од стране ученика позитивно је утицало на систематизацију знања о алгебарским структурама и на постигнућа ученика (са посебним способностима за математику) из математике (конкретно линеарне алгебре и аналитичке геометрије). Овај резултат имплицира да наставници који раде са ученицима са посебним способностима за математику треба озбиљно да размисле о организовању часова математике на којима ће ученици систематизовати и продубити своја теоријска знања креирањем мапа ума.

Кључне речи: мапе ума, ученици са посебним способностима за математику, алгебарске структуре, настава математике.

Olivera D. Cekić-Jovanović
University of Kragujevac
Faculty of Education in Jagodina

Nenad Z. Gajić
Primary School "Anta Bogićević"
Loznica

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ATTITUDES OF PRIMARY SCHOOL TEACHERS ABOUT THE IMPORTANCE, PLACE AND ROLE OF MODERN TECHNOLOGY AND MATHEMATICS IN STEAM EDUCATION

Abstract: The natural connection between mathematics, natural and technical sciences should be present and visible through very intense correlations between the respective school subjects. In many educational systems, the field of STEAM is recognized, which implies a holistic approach and integration of natural sciences (biology, chemistry, physics, physical geography), technical and engineering sciences (electrical engineering, mechanical engineering, construction, hardware and software engineering), art and mathematics. The aim of the paper is to examine the views of elementary school teachers (N = 160) about the importance, place and role of modern technology and mathematics in STEAM education. The study discussed the basic theoretical starting points, possibilities and challenges of applying modern technology and mathematics in integrative STEAM teaching. Also, teachers' attitudes were examined using the research survey technique, using as a research instrument an anonymous questionnaire created in the Google Forms web application as a five-point Likert-type scale. The research results confirm the positive attitudes of teachers about the importance of modern technology and mathematics in STEAM education. Also, the research results confirm that primary school teachers have positive attitudes toward and often apply modern technology and mathematics in STEAM classes for the preparation of materials, research activities and individualization of the teaching process. However, to a lesser extent, they attend professional development seminars focused on the application of modern technology in STEAM integrative teaching; there is a possibility to improve this important segment of education.

Keywords: STEAM education, integrative approach, teachers' attitudes, educational technology, mathematics, Serbia.

INTRODUCTION

The integrality of teaching means the realization of the requirement (principle) that all elements of the teaching process – content, psychology, cognition, sociology and organization – are functionally connected and form a harmonious

o.cekicjovanovic@gmail.com

whole (Vilotijević 2016). Teaching based on this approach can be very stimulating and motivating for students. Content that is interconnected contributes to knowledge that is complete, valuable and usable (Spremić 2007). Integrative teaching is teaching in which the boundaries between different subjects or disciplines are erased or partially imperceptible. This type of teaching makes meaningful connections between similar aspects of different disciplines. Disciplines mutually integrate, permeate and synthesize into a new whole that is larger and more significant than the simple sum of its constituent elements (individual subjects, disciplines) (Spremić 2007). The integrative approach primarily contributes to building a comprehensive picture of reality that students encounter in real life and helps to develop a divergent way of thinking and originality (Jovanovic, Kovcic 2017).

Vilotijević states that Gestalt theory (main representatives Wertheimer, Kafka and Keller) provided a good theoretical basis for integrative teaching. In this theory, the main point suggests that psychic processes cannot be broken down into small parts, since organization and integrity are the most important features of psychological processes, and they are lost through atomization (Vilotijević 2016). The effort to make the idea of integrative teaching a practical reality is present in many advanced pedagogical movements and directions during the 20th century. Ideas of integrative teaching can be found in the concept of active school from A. Ferriera, the project-method from J. Dewey, the idea of exemplary teaching, and the pedagogical-methodical ideas of Sreten M. Adžić. Cekić-Jovanović and Mihajlović observe that Adžić, through his examples from practical experience, encourages students to consider different perspectives, to connect facts, to think critically and creatively about ideas, to process and learn in different ways, to experience contents and to create their own original, individual works (Cekić-Jovanović, Mihajlović 2018). Most authors talk about three forms of teaching integration: full, partial and block. A complete form of integration means combining different teaching contents into a single course. Partial integration is when certain chapters that have similarities are selected from the teaching material, so they are processed together. In the block mode of integration, autonomous blocks are created that are independently programmed or parts of the joint program are separated for integrative processing. The levels of integration are intra-subject, inter-subject (use of inter-subject links) and inter-system integration (combining the contents of different subjects into a whole) (Drobnjak 2007).

STEAM is a teaching method that applies meaningful science, technology, engineering, art, and math content to solve real-world problems through hands-on learning activities and creative design (Bošković, Lalić, Milić 2020). It was created as a solution to the main shortcoming of STEM, which is the development of creative thinking and applied art in solving problems. The acronym STEM was created in the 1990s by the American National Science Foundation in order to better promote the integrative learning of science, technology, engineering and mathematics (Sanders 2009). There is a large number of studies that confirm the advantages and didactic value of STEAM education (Gunčaga, Kopczynski 2019; Gutschank 2019; Đorđević, Kopas-Vukašinović, Mihajlović 2019; Stohlmann, Moore, Roehrig 2012).

Integrated STEAM education should be viewed as a space for students to apply their knowledge of disciplines to create products and/or solve problems that can be made or addressed using engineering principles (Brackley, Howell 2019). STEAM creates a safe environment for students to express and experience their ideas, which encourages them to think outside the box (Strutynska, Umryk 2019). A US News article reported that Andover High School is teaching geometry through art. Mathematics and art teachers used the game "scavenger hunt" in a local museum to make students understand that projective geometry is the same thing as perspective in art (Bošković, Lalić, Milić 2020).

Strategic documents and laws dealing with education in Serbia emphasize the importance of an integrative approach. The Law on Basic Education and Upbringing (*Official Gazette of RS*, No. 55/2013, 101/2017, 10/2019 and 27/2018 – State Law) defines the basic goals of basic education and upbringing, which foresees the development of key competencies for lifelong learning and cross-curricular competence in accordance with the development of modern science and technology, as well as the development of creative abilities, critical thinking, motivation to learn, ability to work in a team, and ability to take initiative and express one's opinion. The strategy for the development of education in Serbia until 2020 envisages the development of students' divergent thinking, creative abilities, creative potentials and the acquisition of higher-quality, practically applicable knowledge from various fields, and at the same time aspires towards cross-curricular planning and linking of teaching content. The application of innovative ways and methods of teaching represents a good basis for the introduction of the STEAM model into the formal framework of the educational system. The application of the STEAM model as a concrete action for the development of education is in accordance with the vision of the future state of the education system in Serbia. According to the Education Development Strategy in Serbia until 2020, primary education and upbringing is a good and stimulating environment in which students master quality knowledge and skills that can be interconnected and applied in further education and in everyday life (Education Development Strategy in Serbia 2020). The Rulebook on the teaching and learning plan for the first cycle of primary education recommends that integration, correlation and connection of the contents of different subjects should be carried out wherever possible in order to enable the complete development of the student's personality, the acquisition of quality knowledge, and the development of divergent and critical thinking (*Official Gazette* 2017, 2018, 2019a, 2019b, 2020). In the education of teachers, as key participants in the education system, special emphasis should be placed on strengthening their competencies for teaching and teaching methodology (K1) and competencies for teaching and learning (K2) (Regulation on standards of competencies for the profession of teachers and their professional development, 2011).

The natural connection between mathematics, natural and technical sciences should be present and visible through very intense correlations between the respective school subjects. In many educational systems, the field of STEAM is recognized, which implies a holistic approach to the presentation of natural sciences (biology, chemistry, physics, physical geography), technical and engineering sciences (electrical engineering, mechanical engineering, construction, hardware and software engineering), art and mathematics. Establishing correlations of mathematics teaching with other STEAM disciplines can contribute to stronger student motivation and a deeper understanding of all areas that are integrated (Maass, Geiger, Ariza, Goos 2019). The advantages of integration are more than obvious. For example, a large number of problems in the field of computing cannot be solved without adequate mathematical knowledge. Conversely, the use of computers makes it easier to solve many mathematical problems and can contribute to a better understanding of certain mathematical concepts (primarily through the introduction of visualization and experimental methods in mathematics teaching) (Marić 2020).

Lipkovski claims that mathematics is essentially used in all natural and technical sciences. Ever since the age of Copernicus, Galileo and Newton, fundamental and new mathematical concepts have been created and developed on the one hand as a means for the progress of natural sciences, while on the other hand, every natural science intensively uses already existing mathematical methods in its development. The best example of this is the general theory of relativity, in which Einstein used the already existing theory of differential geometry. In the words of the German philosopher Kant, there can only be as much real science in any natural science as there is mathematics in it (Lipkowski 2020).

The correlation of mathematics and natural sciences in elementary school is best seen starting from the 7th grade with the teaching of physics, percentages and proportions in chemistry, although before 7th grade the mathematical concepts of scale and proportion also appear in physical geography (5th grade).

The example of Fibonacci rabbits (i.e. obtaining its sequence and the value of the golden section) reflects the correlation of mathematics and biology in the best possible way. It should certainly be mentioned that, in addition to the natural sciences, mathematics also occurs in other spheres, such as music, fine arts, and literature. When children learn rhythm and notes, they simultaneously learn division, fractions and proportions. Some studies have shown that people who know math are better at playing the piano (Nemirovsky 2013). Fine art relies heavily on symmetries, perspective, and projective geometry. Various literary works describe mathematical concepts in an even clearer form than their mathematical definition. The famous Goethe and his Faust describe the magic square through verses; Jules Verne in *From the Earth to the Moon* gives very precise definitions of the parabola and the hyperbola. By learning mathematics, students acquire important skills needed for later computer and programming careers. On the one hand, mathematical education fosters the acquisition of concrete knowledge and develops a mathematical apparatus that has direct applications for solving practical tasks, and on the other hand, mathematical education contributes to the development of general cognitive abilities and the development of an appropriate approach to solving problems that is useful in all IT disciplines (Sevimli, Ünal 2022). Some of the concepts that are developed in the teaching of mathematics and are very important for the overall development of the student's personality, the acquisition of quality knowledge, and the development of divergent and critical thinking include: algorithmic procedures, mathematical logic, decomposition of problems into simpler problems, formal language, calculation and evaluation of values, computer graphics and geometry, and data analysis and processing (Marić 2020). Correlation of mathematics with programming and informatics enables most routine tasks to be automated, so it is possible to solve more complex problems. For example, with the use of a computer, it is possible to solve systems of linear equations with several dozen unknowns and equations, which can be used to build a fairly accurate model of a real-life problem. Problem-oriented teaching insists on the practical applicability of introduced concepts to solve concrete examples. For example, instead of the mechanics of calculating determinants by hand, it becomes much more important that the student can recognize that the determinant is a measure of area (i.e., volumes of the parallelepiped formed by its column vectors) and that the calculation of areas or volumes can then be reduced to the problem of calculating determinants. All of this poses a much greater challenge to students (and teachers) than in the case when only abstract tasks are solved, isolated from the general context of application and specifically prepared only in order to practice some concrete technique. This certainly means that weaker students will have difficulties in such activities. On the other hand, working in a team, peer teaching and the awareness that concrete problems are solved as part of formal education, the meaning of which students immediately understand, can lead to greater student motivation and thus to better results (Marić 2020).

RESEARCH METHODOLOGY

The aim of the research was to examine the views of elementary school teachers about the importance, place and role of modern technology and mathematics in STEAM education. In accordance with the set goal, research tasks were formulated.

– Examine whether and to what extent elementary school teachers apply integrative STEAM teaching;

– Examine whether and in what way elementary school teachers apply modern technology and mathematics in STEAM classes;

– Examine the views of elementary school teachers on the importance of modern technology and mathematics for STEAM education;

– Examine the views of elementary school teachers about the role of modern technology and mathematics in STEAM education;

– Investigate whether and to what extent primary school teachers attend professional development seminars focused on STEAM integrative teaching.

The descriptive method, the survey technique, was used in the research, and an anonymous questionnaire was created in the Google Forms web application as a research instrument.

The questionnaire consisted of three segments. The first part includes general information about the respondents (gender, level of professional education, years of work experience, place of school where they work, subject they teach), the second part of the questionnaire is a five-point Likert-type scale and includes 19 statements, and the third part of the questionnaire consists of 2 open-ended questions. The questionnaire was created by the authors of the paper based on previously studied literature. The value of the Cronbach alpha coefficient is 0.800 which indicates good reliability of the research instrument.

The sample of respondents is random, and the population that participated in the research are primary school teachers on the territory of Serbia. The electronic questionnaire was distributed via social networks and e-mail addresses that are in the database of Serbian teachers' associations. Analyzing the structure of the sample, our data showed that the most respondents work in city schools (99), then in rural schools (38) and the least in suburban schools (23). 119 subject teachers and 41 classroom teachers participated in the research, of whom 120 (75%) were women and 40 (25%) were men. Of the 160 respondents, there is an approximately equal number of those with 1–10 years of service (N = 45), 11–20 years (N = 46) and 21–30 years (N = 44) of service, while the smallest number is those with more than 30 years of service (N = 25).

When it comes to the data related to the subject area that the respondents teach, most of them (25%) teach social sciences (languages, history), all subjects 23.1%, mathematics 21.3%, natural sciences 15.6% (physics, chemistry, biology, geography), 7.5% of respondents teach technical (technique and technology, technical education, informatics) and 7.5% teach art and skills (music culture, art culture, physical education)

RESEARCH RESULTS

The first group of questions refers to the ability of teachers to implement STEAM teaching and the frequency of application of integrative teaching.

The largest share of respondents (57%) often functionally connect the contents of different subjects in teaching practice; 40% do it sometimes, and 3% of respondents rarely or never connect the contents of different subjects in teaching practice (M = 4.50, SD = 0.691). Also, most of the respondents (67%) claim that they connect situations from real life with the contents of science, technology, art and mathematics and use them in classes as examples for learning (M = 4.59, SD = 0.704). 57% of respondents partially agree and 19% of respondents completely agree that they often design and plan research activities that integrate the contents of different subjects (M = 3.74, SD = 1.043).

However, when it comes to the frequency of applying an integrative approach and connecting the content of different subjects according to the STEAM model, the largest number of respondents partially agree (54%) and 25% completely agree. As many as 21 respondents (13%) do not know whether they use an integrative approach in teaching practice and connect the contents of different subjects according to the STEAM model (M = 3.96, SD = 0.846).

Based on the obtained values of Levene's test of equality of variance and corresponding indicators of significance, we can conclude that there are statistically significant differences between classroom teachers and subject teachers when they respond to the items of the scale related to the ability of teachers to implement STEAM teaching and the frequency of application of integrative teaching, i.e. that classroom teachers are more qualified to implement STEAM teaching and apply integrative teaching more often than subject teachers (Tables 1 and 2). The results are in agreement with research from 2014 that showed that integration is easier to achieve in classroom teaching, because the material in classroom teaching is not as strictly differentiated as in subject teaching, which facilitates the application of this modern teaching model. "The implementation of integrative teaching in subject teaching, on the other hand, is hampered by excessive plans and programs. Integration is also made more difficult by the fact that it requires coordination between two or more teachers" (Adamov, Olić, Halaši 2014).

Table 1. Descriptive statistics

	Professional qualification	N	Mean	Std. Deviation	Std. Error Mean
Often in teaching practice, I functionally connect the contents of different subjects.	classroom teachers	41	4.68	0.471	0.074
	subject teachers	119	4.44	0.744	0.068
Very often I connect situations from real life with the contents of science, technology, art and mathematics.	classroom teachers	41	4.76	0.435	0.068
	subject teachers	119	4.53	0.768	0.070
In my teaching practice, I often use an integrative approach and connect the contents of different subjects based on the STEAM model.	classroom teachers	41	4 22	0.525	0.082
	subject teachers	119	3.87	0.916	0.084
I often design and plan research activities that integrate the contents of different subjects.	classroom teachers	41	4.05	0.773	0.121
	subject teachers	119	3.63	1.104	0.101
I often conduct research activities in classes that integrate the contents of different subjects.	classroom teachers	41	4.07	0.721	0.113
	subject teachers	119	3.47	1.241	0.114

Table 2. Independent Samples Test

		Levene's Test		t-test for Equality of Means						
		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	95% Confidence	
									Lower	Upper
IT1	EVA	5.300	0.023	1.982	158	0.049	0.246	0.124	0.001	0.491
	EVNA			2.452	110.559	0.016	0.246	0.100	0.047	0.445
IT2	EVA	9.076	0.003	1.791	158	0.075	0.227	0.127	-0.023	0.477
	EVNA			2.317	123.793	0.022	0.227	0.098	0.033	0.420
IT3	EVA	6.406	0.012	2.286	158	0.024	0.346	0.151	0.047	0.644
	EVNA			2.944	122.366	0.004	0.346	0.117	0.113	0.578
IT4	EVA	11.605	0.001	2.244	158	0.026	0.419	0.187	0.050	0.787
	EVNA			2.657	99.309	0.009	0.419	0.158	0.106	0.731
IT5	EVA	32.447	0.000	2.940	158	0.004	0.603	0.205	0.198	1.007
	EVNA			3.766	120.714	0.000	0.603	0.160	0.286	0 919

EVA – Equal variances assumed; EVNA – Equal variances not assumed

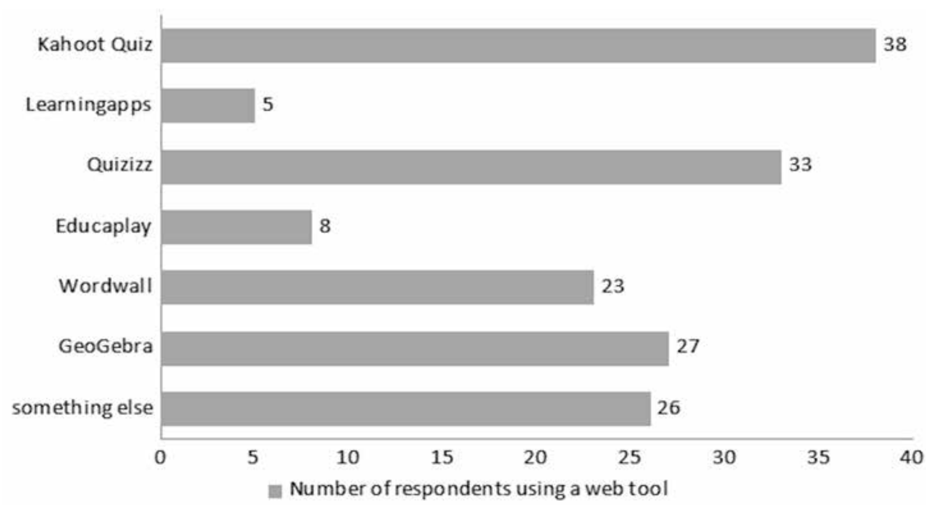
In the continuation of the questionnaire, we examined whether and in what way elementary school teachers apply modern technology and mathematics in STEAM classes. A small number of respondents (about 5%) declared that they do not use modern technology to prepare the materials they use in integrative teaching classes. 150 respondents agree that they use modern technology during integrative teaching.

Table 3. Descriptive statistics

	N	Mean	Std. Deviation
I use modern technology to prepare the materials I use in STEAM classes.	160	4.33	0.716
During STEAM lessons, I often use modern technology.	160	4.29	0.756
During STEAM classes, I often use math content.	160	4.31	0.736
Valid N (listwise)	160		

Almost all teachers who use modern technology for preparing materials (95%) also use modern technology during integrative teaching (94%), which can be seen in the following graphic. Graphic 1 presents the web tools that teachers most often use during integrative teaching.

Graphic 1. The web tools that teachers most often use during integrative teaching.



Respondents who answered "something else" to the previous question had the opportunity to write down which web tool they use during integrative teaching. Among the answers the following appeared most often: *Google Questionnaire* (3 times), *Phet Colorado*, *Genialy*, *TinkerCad* once each, *MakeCode*, *Pintar Virtualab*, *Circuit Virtualab*, *e-classroom*, *Prezi*, *Jigsaw puzzle*, and *Settera*. Four respondents answered that they do not use any web tool.

Of the web tools, mathematicians most often use the *GeoGebra* tool (27); social science teachers most often use the tools that can be used to create quizzes, such as *Kahoot* (38) and *Quizizz* (33); language, arts and literature teachers most often use *Wordwall* (23), while other teachers use the offered web tools with equal frequency.

Of modern devices used during integrative classes in elementary school, smartphones are most often used (61.9%), followed by tablet computers (17.5%) and microbit computers (11.3%).

In most European countries, students and teachers use smartphones as powerful assistants. With high-speed Internet access, knowledge is at one's fingertips. Smartphones have a variety of technological capacities and laboratory applications that even the best-equipped European schools can only dream of. Using smartphones, we can determine geographic position (GPS coordinates), latitude, longitude, altitude, pressure, acceleration, angle of rotation, magnetic fields, and voltage. Also, smartphones contain high-resolution cameras with which we can record a process and manipulate it by speeding up or slowing down playback, enlarging the image, and the like. There are thousands of applications for using the data provided by smartphones, so their application in teaching is considered modern and necessary (Andrade, Richter, Gutschank 2014).

The next part of the results refers to the examination of teachers' views on the importance of modern technology and mathematics for STEAM education.

Regarding the statement that "software tools and applications contribute to the visualization of concepts", almost all respondents completely agree (52.5%) or partially agree (43.1%), which totals to 95.6% of the sample ($M = 4.44$, $SD = 0.707$). A large number of respondents fully agree (38%) or partially agree (52%) that modern educational technologies and differentiated mathematical contents contribute to the individualization of the teaching process ($M = 4.23$, $SD = 0.801$). The result obtained coincides with the result of research conducted in 2010 in Canada that found that educational software designed so that students can use it independently provides an opportunity for teachers to interact 'one-on-one' with those students who need help the most (Means 2010). 42% fully agree and 46% partially agree, which is 88% of respondents ($M = 4.23$, $SD = 0.876$), that modern educational technology includes a greater number of receptors in the learning process and thus contributes to a more complete and efficient understanding. Mathematical contents within STEAM classes develop logical thinking and functional knowledge (40% fully agree and 45% partially agree, which is 85% of respondents ($M = 4.12$, $SD = 0.756$)). 49% completely agree and 38% partially agree ($M = 4.26$, $SD = 0.953$) that students are more motivated to work during teaching activities that integrate the contents of different subjects with the application of modern technology and mathematics. The largest number of respondents (48%) do not agree at all and 28% of respondents partially disagree that the lesson in which the contents of different subjects are integrated is a wasted lesson, because it is impossible to assess the students' knowledge of individual subjects (M = 1.99, SD = 1.231). We can conclude that teachers have positive attitudes about the importance of modern technology for STEAM education.

Based on the obtained values of Levene's tests of equality of variance and corresponding indicators of significance, we can conclude that there are no statistically significant differences between classroom teachers and subject teachers, male and female respondents, teachers working in city, suburban and rural schools, and teachers with differing years of service when they respond to scale items related to the importance of modern technology for STEAM education.

The results of our research are in agreement with the results of previous research that has shown learning using modern technology is more effective than the average lecture, because the concentration of students is maintained at a high level. The aim of the research conducted by Mladenović in 2009 was to teach the same content in two different ways and to compare the achieved results after completing the test. The first group had 54 members and dealt with the material in a traditional way through lectures. The second group also had 54 members and processed the same teaching content through a multimedia course on their computer. The lecture lasted 3 hours in both cases. Of the 54 members who followed the traditional lectures, 31 were unable to reproduce even 20% of the material covered, 15 managed to reproduce 35%, and only 8 managed to reproduce more than 35% of the material covered. In the other group, out of 54 members who worked on the assigned material through multimedia courses, only 11 failed to reproduce at least 20% of the material covered, 6 managed to reproduce 35%, while as many as 37 members managed to reproduce more than 35% of the material covered (Mladenović 2009). In addition, the results coincide with the results of research on the possibilities of improving educational activities at universities by applying an integrative approach within multimedia programmed teaching. The majority of participants recognize multimedia programmed teaching and teaching based on content integration as ways to create practically applicable knowledge and understanding of material more easily. Also, these methods enable them to individualize teaching, i.e. to determine the pace of progress, the source of knowledge and learning according to their own interests (Cekić-Jovanović, Đorđević, Miletić 2018).

Based on the obtained results, we can conclude that primary school teachers have positive attitudes about the importance of modern technology and mathematical content for STEAM education, because modern technology and mathematics contribute to more complete and efficient understanding, visualization of concepts, individualization of the teaching process, student motivation for work and development of logical thinking. These results are similar to those obtained in research by Wei and Matt (2020). By further analyzing the results, we conclude that the largest number of respondents (57%) partially agree and 22% of respondents fully agree that they have acquired basic knowledge and skills for applying an integrative approach to working with students (M = 3.88, SD = 0.921). Also, the majority of respondents (64%) believe that the application of an integrative STEAM approach requires teachers to have constant professional development (M = 4.55, SD = 0.716). However, the majority of respondents (45.6%) did not attend a single professional development seminar on the application of the STEAM model, which may be a consequence of the small number of seminars related to STEAM education in the catalog of professional development programs (http://zuov-katalog.rs/ index.php?action=page/catalog). On the other hand, during the COVID-19 pandemic, there was an expansion of webinars, so one can find various webinars related to STEAM education. In Serbia, such webinars were organized by the STEM Chamber and the Institute for Modern Education.

CONCLUSION

Based on the previously presented results, we can conclude that primary school teachers have seen the importance of connecting related content of different subjects for the overall development of students' personalities and the acquisition of quality knowledge. Also, they often apply integrative STEAM teaching.

It is widely believed that the primary driver of the economy and the creator of new jobs in the future will be innovations resulting from advances in science and engineering. Technology is already replacing workers in some workplaces. Mathematics should be known in order to do science, and science is needed to develop technology. Technology is needed for production – and for that we need engineers. Design should not be neglected either, because products should not only be functional but enjoyable to use too. Therefore, as a result, an increasing number of jobs are likely to require knowledge of STEAM. Establishing the integration of the content of different subjects based on the STEAM model can contribute to better quality learning, functional knowledge applicable in everyday life, development of creative thinking, application of art in solving problems and motivation for work. The research results confirm that elementary school teachers have positive attitudes towards and often apply modern technology and mathematics in STEAM classes for the preparation of materials, research activities and individualization of the teaching process.

One of the main tasks of education in the 21st century is to constantly renew and adapt the skills of lecturers in order to apply new technologies adequately and on a larger scale, because what they themselves do not know, they cannot continue to teach. This means that the scope of their work will increase, but the more dedicated they are to producing quality content and lessons, the better their teaching will be. The adaptation of educational units to new forms of learning must not be neglected either. Education must provide the foundations on which it can later be quickly upgraded, that is, it must learn how to self-upgrade and self-adapt to new technological requirements and to be able to deal with changes. The results of the research show that elementary school teachers attend, to a lesser extent, professional development seminars oriented to the application of modern technology in STEAM integrative teaching; there is an opportunity to improve this important segment of education.

Therefore, we can conclude that the majority of elementary school teachers have seen the important role of modern educational technology and mathematics in STEAM education and have positive attitudes regarding that teaching model.

REFERENCES

Adamov, Olić, Halaši (2014): J. Adamov, S. Olić, T. Halaši, Multidisciplinarni učenički projekti – primer integrisanja sadržaja u okviru teme „Zdrava ishrana", u: S. Cvjetićanin (ur.), *Miniprojekti u nastavi integrisanih prirodnih nauka i matematike 2*, Sombor: Pedagoški fakultet.

Bošković, Lalić, Milić (2020): D. Bošković, D. Lalić, B. Milić, *Upotreba tehnologije u edukaciji*, Novi Sad: Fakultet tehničkih nauka.

Brackley, Howell (2019): S. Brackley, J. Howell, The Next Chapter in the STEM Education Narrative: Using Robotics to Support Programming and Coding, *Australian Journal of Teacher Education*, IV/4, 51–64.

Vilotijević, Vilotijević (2016): M. Vilotijević, N. Vilotijević, *Modeli razvijajuće nastave II*, Beograd: Univerzitet u Beogradu, Učiteljski fakultet.

Gunčaga, Kopczynski (2019): J. Gunčaga, T. Kopczynski, Supporting Mathematical and Digital Competences Useful for STEM Education. E-Learning and STEM Education, *Scientific Editor Eugenia Smyrnova-Trybulska*, 409–419.

Gutschank (2019): J. Gutschank, *Coding in STEM Education*, Berlin, Germany: Science on Stage Deutschland e.V.

Drobnjak (2007): N. Drobnjak, Integrativna nastava, *Obrazovna tehnologija*, 1–2, Beograd: Učiteljski fakultet, 81–91.

Đorđević, Kopas-Vukašinović, Mihajlović (2019): M. Đorđević, E. Kopas-Vukašinović, A. Mihajlović, Teaching competencies of pre-service primary school teachers to use an integrated approach in teaching science, art and mathematics, In: V. Savić, O. Cekić-Jovanović (Eds.), *Professional competences for teaching in the 21st century*, 173–186.

Zakon o osnovnom obrazovanju i vaspitanju, *Službeni glasnik RS*, br. 55/2013, 101/2017, 10/2019 i 27/2018 – dr. Zakon.

Jovanović, Kovčić (2017): M. Jovanović, V. Kovčić, Contemporary Society Challenges of Integrative Teaching. *Sinteze*, 11, 39–71.

Lipkovski (2020): A. Lipkovski, O značaju nastave matematike, *Nastava matematike*, LXV/3-4, 61–66.

Marić (2020): F. Marić, Matematika u obrazovanju budućih IT stručnjaka, *Nastava matematike*, LXV/3-4, 67–75.

Maass, Geiger, Ariza, Goos (2019): K. Maass, V. Geiger, M. R. Ariza, M. Goos, The Role of Mathematics in interdisciplinary STEM education, *ZDM Mathematics Education*, 51, 869–884. Retrieved in November 2022 from https://link.springer.com/article/10.1007/s11858-019-01100-5.

Nemirovsky (2013): R. Nemirovsky, Playing Mathematical Instruments: Emerging Perceptuomotor Integration with an Interactive Mathematics Exhibit, *Journal for Research in Mathematics Education*, XLIV/2, 372–415.

Pravilnik o planu nastave i učenja za prvi ciklus osnovnog obrazovanja i vaspitanja, *Službeni glasnik*, 2017, 2018, 2019a, 2019b, 2020.

Pravilnik o standardima kompetencija za profesiju nastavnika i njihovog profesionalnog razvoja (2011): *Službeni glasnik RS – Prosvetni glasnik*, br. 5.

Sanders (2009): M. Sanders, STEM, STEM Education, STEM Mania, *Technology Teacher*, 68, 20–26.

Sevimli, Ünal (2022): E. Sevimli, E. Ünal, Is the STEM Approach Useful in Teaching Mathematics? Evaluating the Views of Mathematics Teachers, *European Journal of STEM Education*, VII/1, 1–11. Retrieved in November 2022 from https://files.eric.ed.gov/fulltext/EJ1341387.pdf.

Strutynska, Umryk, (2019): O. Strutynska, M. Umryk, Learning Startups as a Project Based Approach in STEM Education. E-Learning and STEM Education, *Scientific Editor Eugenia Smyrnova-Trybulska*, 529–555.

Spremić (2007): A. Spremić, Integrativna nastava, *Obrazovna tehnologija*, 3, 36–48.

Stohlmann, Moore, Roehrig (2012): M. Stohlmann, T. J. Moore, G. H. Roehrig, Considerations for Teaching Integrated STEM Education, *Journal of Pre-College Engineering Education Research,* II/1, 49-63.

Strategija razvoja obrazovanja u Srbiji do 2020. godine (2012): *Službeni glasnik RS*, br. 107.

Strategija razvoja obrazovanja i vaspitanja u Republici Srbiji do 2030. godine (2021): *Službeni glasnik RS*, br. 63.

Strategija razvoja veštačke inteligencije u Republici Srbiji za period 2020–2025. godine, *Službeni glasnik RS*, br. 96/19.

Strategija naučnog i tehnološkog razvoja Republike Srbije za period od 2021. do 2025. godine „Moć znanja” (2021): *Službeni glasnik RS*, br.10.

Strategija razvoja digitalnih veština u Republici Srbiji za period od 2020. do 2024. godine, *Službeni glasnik RS*, br. 21/20.

Cekić-Jovanović, Mihajlović (2018): O. Cekić-Jovanović, A. Mihajlović, Integrativni pristup u pedagoško-metodičkim idejama Sretena M. Adžića, *Uzdanica*, XV/2, 95–107.

Cekić-Jovanović, Đorđević, Miletić (2018): O. Cekić-Jovanović, M. Đorđević, A. Miletić, Possibility of Improving Educational Activities at Universities by Applying Integrative Approach within Multimedia Programmed Teaching, *Innovative Teaching Models in the System of University Education: Opportunities, Challenges and Dilemmas*, Jagodina: University of Kragujevac, Faculty of Education; Koper: University of Primorska, Faculty of Education, 49–61.

Wei, Maat, (2020): W. K. Wei, S. M. Maat, The Attitude of Primary School Teachers towards STEM Education, *TEM Journal*, 9(3), 1243–1251. Retrieved in November 2022 from https://www.temjournal.com/content/93/TEMJournalAugust_1243_1251.pdf.

Оливера Д. Цекић-Јовановић
Универзитет у Крагујевцу
Факултет педагошких наука у Јагодини

Ненад З. Гајић
Основна школа „Анта Богићевић”
Лозница

ЗНАЧАЈ, МЕСТО И УЛОГА САВРЕМЕНЕ ТЕХНОЛОГИЈЕ И МАТЕМАТИКЕ У STEAM ОБРАЗОВАЊУ

Резиме: Природна повезаност између математике, природних и техничких наука треба да буде присутна и видљива кроз веома интензивне корелације између одговарајућих школских предмета. У многим образовним системима препозната је област STEAM која подразумева холистички приступ и интеграцију природних наука (биологије, хемије, физике, физичке географије), техничких и инжењерских наука (електротехнике, машинства, грађевине, хардверског и софтверског инжењерства), уметности и математике. Циљ рада је испитати ставове наставника у основној школи (N = 160) о значају, месту и улози савремене технологије и математике у STEAM образовању. У студији су размотрена основна теоријска полазишта, могућности и изазови примене савремене технологије и математике у интегративној STEAM настави. Такође, испитани су ставови наставника применом истраживачке технике анкетирања, а као инструмент истраживања креиран је анонимни упитник у веб-апликацији *Гугл Формс* као петостепена скала Ликертовог типа. Резултати истраживања потврђују позитивне ставове наставника о значају савремене технологије и математике у STEAM образовању. Такође, резултати истраживања потврђују да наставници основних школа имају позитивне ставове и често примењују савремену технологију и математику у STEAM настави и то за припрему материјала, истраживачке активности и индивидуализацију наставног процеса. Међутим, они у мањој мери похађају семинаре стручног усавршавања оријентисане на примену савремене технологије у STEAM интегративној настави и постоји могућност да се овај важан сегмент образовања унапреди.

Кључне речи: STEAM образовање, интегративни приступ, ставови наставника, образовна технологија, математика, Србија.

Boris Čulina
University of Applied Sciences Velika Gorica
Department of Mathematics

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WHAT IS MATHEMATICS FOR THE YOUNGEST?

(What an old mathematician learned about mathematics from his granddaughter Nina)

Abstract: While there are satisfactory answers to the question "How should we teach children mathematics?", there are no satisfactory answers to the question "What mathematics should we teach children?". This paper provides an answer to the last question for preschool children (early childhood), although the answer is also applicable to older children. This answer, together with an appropriate methodology on how to teach mathematics, gives a clear conception of the place of mathematics in the children's world and our role in helping children develop their mathematical abilities. Briefly, children's mathematics consists of the world of children's internal activities that they eventually purposefully organize in order to understand and control the outside world and organize their overall activities in it. We need to support a child in mathematical activities that she does spontaneously and in which she shows interest, and we need to teach her mathematics that she is interested in developing through these activities. In doing so, we must be fully aware that the child's mathematics is part of the child's world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it. From the point of view of this conception, the standards established today are limiting and too focused on numbers and geometric figures: these topics are too prominent and elaborated, and other mathematical contents are subordinated to them. Adhering to the standards, we drastically limit the mathematics of the child's world, hamper the correct mathematical development of a child, and we can turn her away from mathematics.

Keywords: preschool mathematics, standards for preschool mathematics, the NCTM standards, the "new mathematics" movement.

Words of caution: My four-year-old granddaughter Nina has been my main motivation and a "collaborator" for the views expressed here. I wrote the views in the deep conviction that they can enable a better mathematical development of children than the established standards, and that as such they are worth sharing. For definiteness, I chose the NCTM standards (NCTM 2000), a very clear and precise document with a lot of value but, in my opinion, limited and improperly balanced content, published by the National Council of Teachers of Mathemat-

boris.culina@vvg.hr

ics – the leading organization of mathematics teachers in the USA and Canada. I will primarily refer to Chapter 4: Standards for Grades Pre-K-2. As far as I know, nothing substantial would have changed in further considerations had I taken some other standards for a reference. I believe that what many teachers and educators do or want to do is in accordance with the conception presented here. However, I am a mathematician with expertise in mathematical logic and the foundations of mathematics, and many years of experience in teaching higher mathematics. I have neither the wider experience nor the expertise in the field of mathematics education of the youngest. If we add that thinking about the mathematics education of the youngest is a sensitive topic where wrong attitudes can have significant consequences, it is inevitable to conclude that the views expressed in this article should be subjected to intensified criticism. Given that I am not an expert in the field, my knowledge of the literature and various theories of children's (mathematical) education is far from systematic. I searched the literature as much as I needed to draw conclusions about the problems that interested me. Such an approach led to a non-systematic use of the literature and a non-systematic connection of the conclusions presented here with the relevant literature. My initial guide was the book (Servais, Varga 1971) that I read a long time ago and which left a deep impression on me, especially Varga's introductory article. His words "Every child, by nature, likes learning just as he likes eating" (page 28) were vividly engraved in me. Most of the students I worked with no longer had that hunger for learning mathematics. For too many of them, this hunger for learning has been replaced by an aversion to mathematics. I have always considered it an unacceptable state of affairs. However, when I entered the world of mathematics learning for the youngest and realized that such a situation exists there, moreover, that it arises there, I experienced it as violence against children. What especially bothers me is hearing that a child is not good at math. In addition to the fact that we should be very careful with such claims, how can we even claim this if we do not properly understand what mathematics is? Rigid standards lead to such unnecessary disqualification of children. I am deeply convinced that changes in the mathematics education of the youngest are necessary and that the time for the changes has come. A sufficiently broad understanding of mathematics is very important here. I hope this article will contribute to such an understanding.

1. INTRODUCTION

Assisting Nina in her mathematical development, I realized that this development is very important for her overall development and that I understand quite well how to teach her mathematics, but, to my surprise, I do not know what mathematics to teach her, although I've been doing math my whole life. I started searching the scientific literature. There I found confirmation of the almost crucial importance of mathematical development at the preschool period for the future mathematical and overall development of a child. For example, we can read (Moss et al. 2016: 154): "Accumulating evidence confirms that children's mathematics learning in the first six years of life has profound, long-lasting outcomes for students in their later years – not only in relation to their future mathematics achievement but also in terms of overall academic success." These studies only confirm for mathematics educators what the creators of early childhood education realized long ago: that the first six years of life are the most important period in a person's development.[1] Furthermore, recent research has shown that children at this age are much more mathematically capable than previously thought. Thus, in (English, Mulligan 2013) editors begin the preface (page 1) with these words: "This edited volume emanated primarily from our concern that the mathematical capabilities of young children continue to receive inadequate attention in both the research and instructional arenas. Our research over many years has revealed that young children have sophisticated mathematical minds and a natural eagerness to engage in a range of mathematical activities. As the chapters in this book attest, current research is showing that young children are developing complex mathematical knowledge and abstract reasoning a good deal earlier than previously thought." Regardless of my experience with Nina, these results did not surprise me at all. It is known that early childhood is a period of exceptional creativity and imagination[2] (if appropriate conditions are ensured for the child), and in the views of mathematics that will be presented below, creativity and imagination are the key elements of mathematical activities, although in general culture these abilities are usually associated with art. I also found out that my teaching of Nina was in accordance with a certain methodology of mathematical teaching of children. This methodology is mostly established and provides satisfactory answers to the question of how to teach children mathematics. In short, *the child's mathematical activities must be part of the child's world – part of her daily activities, part of her play, incorporated into children's stories that she enjoys listening to. Mathematical activities must have their motivation, meaning and value in the child's world, and not from the outside, in the world of adults. In developing mathematical abilities, children must have freedom and not the pressure to achieve pre-established learning outcomes.* I have singled out two of the many quotations that confirm this methodological approach. Tamás Varga (Servais, Varga 1971: 16) writes: "To realize and enjoy the beauty of mathematics, pupils must be given sufficient opportunity for free, playful, creative activity, where each can bring out his own measure of wit, taste, fantasy, and display thereby his personality." Susan Sperry Smith (Smith 2001: 16) writes: "Most experts believe that children's play is the key to mental growth. Time to play and a wide variety of

[1] It was this knowledge that motivated Friedrich Fröbel to design and introduce kindergartens into modern society in the first half of the 19th century.

[2] See, for example, the chapter on creativity in Bilbao (2020).

concrete materials are essential. Children should not be rushed to finish a project or hurried from one activity to another." As Georg Cantor said that the essence of mathematics is in its freedom (Cantor 1883: 19), we could also say that the essence of mathematics education of a child is in her freedom. It is up to us to help to guide children in developing their mathematical abilities, respecting their world and their individuality – which activities and at what stage of his growth attract him – and providing a social environment for free communication and joint action. At the community level, this requires developing the awareness of the importance of children's mathematical development and the willingness of the community to invest money in creating adequate working conditions for educators and teachers. Developing such awareness is especially important because the existing school systems, as far as I know, are generally contrary to this methodology, in theory with their uniformity and evaluation system, and in practice with challenging working conditions for educators and teachers. This methodology is not only related to mathematics education but refers to the overall education of children. Although its roots can already be found in ancient Greece[3], this methodology was developed in the 19th century by the founders of modern education, Pestalozzi, Fröbel, Montessori and many others (see, for example Lascarides, Hinitz 2000).[4]

But what about the question "What math should we teach children?" I was not satisfied with the answers I found. Numbers and geometry? That answer could have been satisfactory until the middle of the twentieth century. Truly, until the middle of the nineteenth century mathematics was described as the science of numbers and (Euclidean) space. The appearance of non-Euclidean geometries which are incompatible with Euclidean geometries but are equally logical in thinking and equally good candidates for the "true" geometry of the world has definitely separated mathematics from the truths about nature. This separation has freed the human mathematical powers, and it has caused the blossoming of modern mathematics. The new views of mathematics have spread into the mathematics community mainly through the works of Richard Dedekind, David Hilbert, Emmy Noether, Van der Waerden and Bourbaki group, and they have become the trademark of modern mathematics. With the end of World War II, it became clear that there was a big discrepancy between modern mathematics which proved to be very important for modern society and mathematics taught in school. The "new mathematics" movement of the 1950s and 1960s, which was the most intense in USA, tried to introduce modern mathematics to school. This movement unfortunately failed, not only because of social circumstances but also because of the one-sided structural-

[3] For example, in Lascarides, Hinitz (2000: 9) we can find: "The Greek idea of childhood is interwoven with play. The Greek word for child is *pais*, and the word for *I play* is *paizo*, both having the same root."

[4] Studying their works, I was personally fascinated by the wealth of educational knowledge they left behind and frustrated by the ignorance of this knowledge in today's wider educational practice.

ist view of mathematics inspired by the Bourbaki group.[5] Thus, for example, the famous American mathematician Marshall Stone, in an article (Stone 1961) in which he very clearly explains the changes that have occurred in mathematics, characterizes modern mathematics "as the study of systems comprising certain abstract elements and certain abstract relations prescribed among them". Stone believes that this must be the backbone of mathematics education. At a symposium organized by The Society for Industrial and Applied Mathematics (Carrier et al. 1962), another famous mathematician Richard Courant clearly identified the dangers of such an approach: "The danger of enthusiastic abstractionism is compounded by the fact that this fashion does not at all advocate nonsense, but merely promotes a half truth. One-sided half-truths must not be allowed to sweep aside the vital aspects of the balanced whole truth." Just as half a dinghy is no longer a dinghy, so half the truth about mathematics is not the truth about mathematics. Unfortunately, we will never know if the reform would have been successful had its creators complemented their program with the "other half" of the truth about mathematics, which includes its content and usability, as well as its origin and development. The reform took place in such a way that the younger the age, the worse the results became. Although the reform failed, its traces remained in modern mathematics education, even of the youngest. For example, although the NCTM standards are dominated by numbers and geometry, there are many structural elements in the elaboration of these themes that were highlighted by the creators of the "new mathematics". Also, additional contents are included: classifying (sets), sorting (equivalence relations), ordering (ordering relations), matching (functions), patterns, chance, change, etc. However, they are mostly subordinated to the numbers and geometry of figures. Even if we single out these contents in relation to numbers and geometry, my feeling was that they still offer a too limited answer to the question of what mathematics to teach children. Thinking about this question, I realized that it is closely related to the question "What is mathematics?" – a question that I have been dealing with all my life. Having thus connected what I was doing in mathematics with the problem of what mathematics to teach Nina, I began to unwind the knot.

In the next section, I briefly describe the philosophy of mathematics that I stand for. In the third section, I present the answer that this philosophy of mathematics gives to the question of what mathematics to teach children, and I compare that answer with the established standards of mathematics education. In the remaining two sections, I highlight some elements that I believe are particularly important in the mathematical development of the youngest and give some comments on the NCTM standards.

[5] A detailed analysis of the „new mathematics" movement can be found in Phillips (2015).

2. WHAT IS MATHEMATICS?

The philosophy of mathematics has not yet given a generally accepted, unambiguous and well-developed answer to the question “What is mathematics?”. Fortunately, mathematics survives quite well without a definitive answer to this question, although the philosophies of mathematics have strongly influenced the development of mathematics. The field of mathematics education is also developing regardless of the lack of a definite answer to the question of what mathematics is. Yet, the answer necessarily affects mathematics education, as do various psychological views on the nature of child development. One should be very careful because wrong or one-sided answers can have negative consequences, as the example of the “new mathematics” movement has shown. Roughly, philosophical answers to the question of what mathematics is can be divided into two groups. According to one group we discover mathematics, according to another we create mathematics. Simply put, the various philosophies of mathematics are divided according to whether natural numbers were discovered or created. Which view we adopt should certainly have an impact on how we teach numbers to children. For example, if numbers exist in a particular world of Plato, then special methods need to be devised to get children into that world and teach them how to discover numbers there. If numbers are created, then we need to show children how to create them. The philosophy of mathematics I stand for has nothing in common with realistic views of mathematics, according to which mathematical objects and mathematical worlds belong to the external world. According to this philosophy of mathematics, the human being and the human community create mathematics, just as they create, for example, works of art. This view of mathematics is close to Hersh’s humanistic philosophy of mathematics (Hersh 1997) and Ernest’s social constructivist philosophy of mathematics (Ernest 1997) and can be considered a certain elaboration and modification of their views in one part. This philosophy encompasses structuralism, constructivism, formalism, and fictionalism in a way that avoids their one-sidedness. It is described in detail in (Čulina 2020). Here I will briefly present it and draw the consequences for the mathematical upbringing of children. As far as I can see, the only source of its one-sidedness may be in not accepting mathematics as part of reality. From my personal teaching experience, I know that looking at mathematics as a free and creative human activity is a far better basis for learning mathematics than looking at it as an eternal truth about some elusive world.

The philosophy I will briefly present here has the same roots as modern mathematics – in the emergence of non-Euclidean geometries that led to the separation of mathematics from truths about reality. According to this philosophy, mathematics is not a science of the truths of the world, but it is a means of discovering those truths; it is human invention whose purpose is to be a tool of our rational cognition and rational activities in general. This purpose significantly influences its design and determines its value. Dedekind summed it up nicely with the example of numbers (Dedekind 1888): "[…] numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things". Mathematics is a process and result of shaping our intuitions and ideas about our internal world of activities into thoughtful models which enable us to understand and control better the whole reality. By "internal world of activities" I mean the world that would disappear if we became extinct as a species and that consists of activities over which we have strong control and which we organize and design by our human measure (e.g., movements in space, grouping and arranging small objects, writing on paper, talking, painting, playing music, etc.). It is from these concrete activities that the idea of an idealized mathematical world (model, theory) emerges, the world that expands and supplements the internal world of activities. Mathematical truths are not truths about the external world but specifications (formulations) of a mathematical world. Unlike scientific theories that are true or false about something, mathematical theories are good or bad for something.

For simplicity, I will explain this process of creating a mathematical model on the paradigmatic example of natural numbers. In his book (Mac Lane 1986) Sounders Mac Lane describes this process on a multitude of examples. Natural numbers are the result of modelling our intuition about the size of a collection of objects. This intuition stems from comparing smaller collections from our everyday world of internal activities. We measure a collection by process of counting, and natural numbers are objects created for counting. To start counting we must have the first number, to associate it to the first chosen object in the collection. To continue counting, after each number we must have the next new number, to associate it with the next chosen object in the collection. Conceptually, there is no reason to sort out some particular objects as natural numbers. Merely for the needs of calculation we sort out a particular realization, in the past through collections of marbles on an abacus, and today sequences of decimal numerals on paper and of bits in a computer. It means that for counting it is not important how numbers are realized, but only the structure of the set of natural numbers which enables us to count is important. It seems that they exist in the same way as chess figures, in the sense that we can always realize them in some way. However, the structure of natural numbers, as opposed to the structure of chess, brings an idealization. To be always possible to continue counting, each natural number must have the next natural number. Therefore, there are infinitely many natural numbers. So, although we can say for small natural numbers that they exist in some standard sense of that word, the existence of big natural numbers is in the best case some kind of idealized potential existence. Thus, we come to the idea of an idealized world of numbers that we cannot fully construct. We can only specify that world in a certain language. In that language we have names for numbers, predicate expressions for relations between numbers, and function expressions for operations between numbers. Language is primarily important as a carrier of abstraction. It separates what is important to us for numbers (first number, successor, predecessor, comparison, etc.) from what is not important (e.g., size of marbles if we use them for numbers, or font of decimal numbers if we use them for numbers). I would like to point out here that numbers are not abstract, but that we do abstraction with the help of language! The same is true for other mathematical objects. Furthermore, we specify the properties of this idealized world by certain claims of the language itself that we can axiomatically organize. This is necessary because, although we have the interpretation of the language, the recursively defined truth value of sentences is not a computable function due to the infinite domain of the interpretation. The axioms of natural numbers are neither true nor false, just as the axioms that would describe the game of chess would be neither true nor false. They are a means of specifying our ideas about natural numbers into a coherent mathematical model. It is the same with other mathematical models. Ultimately, they are always a combination of a partial interpretation in the world of our internal activities and additional specification by means of statements (axioms) of a language – a language by which we also achieve the necessary abstraction. The interpretation itself can be significant only up to isomorphism, as is the case with natural numbers, where only their structural properties in the counting process are important to us. But this is not always the case, and that is why the structural approach is one-sided. The best example of this is Euclidean geometry. It stems from our intuition about the space of our everyday activities. It is shown (Čulina 2018) how the idealization of these activities leads to Euclidean geometry. Thus, Euclidean geometry has a prominent interpretation in the world of our internal activities and is not determined structuralistically, up to isomorphism. Thoughtful modelling of other intuitions about our internal world of activities leads to other mathematical models. First, there is a not so big collection of primitive mathematical models ("mother structures" in Bourbaki's terminology (Bourbaki 1950)) that model the basic intuitions about our internal world of activities: intuition about near and remote (topological and metric structures), about measuring (spaces with measure), about straight and flat (linear spaces), about symmetry (groups), about order (ordered structures), etc. We use them as ingredients of more complex mathematical models. The complex mathematical models enable us to realize some simple and important mathematical ideas (for example, we use normed linear spaces to realize an idea of the velocity of change) or they have important applications (like Hilbert spaces which, among other things, describe the states of quantum systems). Furthermore, various mathematical models are interwoven. We express these connections by corresponding mathematical models too: these are secondary mathematical models that model how to build and compare structures (set theory and category theory) and in what language to describe them (mathematical logic). However, regardless of the complexity of the world of modern mathematics, its essence is an inner organization of rational cognition and rational activities in general based on the modelling of intuition about the world of our internal activities.

3. WHAT IS MATHEMATICS FOR THE YOUNGEST?

From this philosophical point of view on the nature of mathematics follows the answer to the question “What mathematics should we teach preschool children?”. Just as the world of internal activities of adults is a source of mathematics for adults, so the world of internal activities of children is a source of children’s mathematics. It manifests itself most expressively and develops best in children’s play, being the key element of the play. Often the purpose of children’s play is to understand the outside world (“let’s play with dolls”, “let’s play cooking”, etc.). When such a purpose is added to the play, then in the world of children, as well as in the world of adults, we have a mathematical model of a phenomenon. Children’s stories themselves can be understood as mathematical models of certain phenomena. The Witch, for example, represents evil, Hansel and Gretel goodness, which, aided by wisdom, defeats evil and forgives the deceived (their father) but not the incorrigibly evil (The Witch and their stepmother). Here art and mathematics are almost indistinguishable.[6] The lesson is clear: the more play there is, the more math there is in the children’s world. In addition to play, children develop mathematical skills whenever they try to organize their daily lives with the help of adults: arrange their toys and clothes, plan what they will do, etc. Thus, *children’s mathematics consists of the world of children’s internal activities that they eventually purposefully organize in order to understand and control the outside world and organize their overall activities in it.*[7] *We need to support a child in mathematical activities that she does spontaneously and in which she shows interest, and we need to teach her mathematics that she is interested in developing through these activities.* This answer to the question “What mathematics should we teach preschool children?” is completely in harmony with the methodological answer to the question “How do we teach children mathematics?”, which is described above. Briefly, *a child’s mathematics is part of a child’s world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it*. I believe these answers, though general, give a clear conception of the place of mathematics in the children’s world and our role in helping children develop their mathematical abilities. Having a clear conception is one of the key prerequisites to assist parents, educators, and teachers to successfully help the youngest in their mathematical development. In what follows, I will single out elements that are more mathematical in the sense that they empower children for more effective control of reality. Usually only these isolated elements are considered mathematics for children, as in the NCTM standards. In this way, the orientation and awareness

[6] Art and mathematics thus have the same source. Later they are differentiated by purpose, but this connection remains. That’s why many, including me, believe that mathematics is, among other things, also a kind of art.

[7] Thereby, it is neither necessary nor possible in the child’s current activities to strictly distinguish between what is and what is not mathematics.

that children's mathematics encompasses much more than these isolated elements is lost. Much more attention should be paid to the free child's play, stories, and organization of the child's daily life as part of his mathematics and the development of appropriate content. In the NCTM standards, this is not considered mathematics but an environment in which mathematical elements should be inserted. Thus, if we adhere to the NCTM standards then we limit the mathematical development of a child. The lack of recognition of these activities in math standards does not necessarily prevent the correct mathematical development of the child as these activities are naturally present in the development and upbringing of a child. However, the lack of recognition can lead to the fact that the environment, including the child herself, believes that she is not inclined to mathematics, even though she is. As for the elements that are more mathematical (in the sense described above), they of course include natural numbers to control quantities and geometry to control spatial activities. However, my conclusion, which I will explain below, is that the NCTM standards neither cover all the essential mathematical elements nor properly distribute attention to those elements they cover. My main criticism is that in preschool and primary school education numbers are too prominent and too elaborate and that other mathematical activities are unnecessarily subordinate to them, while in geometry too much importance is given to figures and bodies that reflect the world of adults more than the world of children. Reading the NCTM standards we can easily be convinced of this dominance of numbers and geometric figures. In the introductory chapter the following is written about the role of numbers in the mathematics education of children (page 32): "All the mathematics proposed for prekindergarten through grade 12 is strongly grounded in number. The principles that govern equation solving in algebra are the same as the structural properties of systems of numbers. In geometry and measurement, attributes are described with numbers. The entire area of data analysis involves making sense of numbers. Through problem solving, students can explore and solidify their understandings of number. Young children's earliest mathematical reasoning is likely to be about number situations, and their first mathematical representations will probably be of numbers." Especially for the youngest age, the following is written (page 79): "The concepts and skills related to number and operations are a major emphasis of mathematics instruction in prekindergarten through grade 2." The introductory part on geometry begins with the following text (page 41): "Through the study of geometry, students will learn about geometric shapes and structures and how to analyze their characteristics and relationships." Especially for the youngest age, the following is written (page 97): "Pre-K-2 geometry begins with describing and naming shapes." My goal is to show that with such an approach we are drastically limiting the mathematics of the children's world, hampering the natural mathematical development of a child, and risking that a child develops an aversion to mathematics. Indeed, as Tamás Varga (Servais, Varga 1971: 21) has pointed out, the real question is not at what age to teach a given area of mathematics but what to teach from *every* area of mathematics at a given age. *To answer in more detail the question "What of numbers and geometry, as with any other elements of mathematics, to teach the youngest?", we must take great care that it is not mathematics that belongs to our adult world but mathematics that fits into the children's world.* A detailed and complete answer to this question is, of course, beyond the scope of this article and beyond my capabilities. Finally, it is an answer that necessarily changes over time. Below I will highlight some elements that I consider to be particularly important in the mathematical development of preschool children and make some remarks on the NCTM standards.

4. PRIMARY MATHEMATICAL ELEMENTS

4.1. SETS, RELATIONS, AND FUNCTIONS

The building blocks of modern mathematics are sets, relations, and functions. They are used to build, connect, and compare mathematical structures. That is why the creators of the "new mathematics" believed that these elements must be at the very basis of mathematics education. So, they thought that the teaching of the youngest should start with these elements – which proved unsuccessful. The reason is simple to me: these concepts are foreign to the children's world. The concept of set derives from the grouping and classification of objects. However, while it is natural for children to work with concrete objects, it is not natural for them to work with abstract sets of objects. For example, a child will naturally group blue objects. She will be able to tell which object is blue, but she will have a problem if we ask her what it means "to be blue". In other words, she knows how to use the predicate "to be blue" but she cannot say what it means "to be blue". It is the same with other predicates. A child learns to use them correctly in classifying objects, but they themselves are not the object of her activities. We could go further: a child learns to use language, and with the help of language to articulate and structure her activities, but language itself is not the object of her activities at that age. Reflection on language and thinking comes mostly later. Since sets are determined by one-place predicates, relations by multi-place predicates and functions by function expressions, using language the child uses sets, relations, and functions in working with objects, but they are not the objects of her activities. Nina will talk about objects on the table and not a set of objects on the table. She will say that Ezra and Nina are cousins, but she will certainly not say that they are in a relationship of "being a cousin". She will say that Anja is Ezra's mother but not that Anja is a value of the "mom of" function applied to Ezra. *Instead of telling them about sets, relations, and functions, we need to teach children to perceive and construct concrete sets, relations, and functions.* This is what of these concepts, in my opinion, should be taught at this age. And the children's world is full of concrete examples of sets, relations, and functions. Children learn sets by grouping and classifying objects, relations by comparing objects, and functions through concrete actions over objects. All these activities are included in the NCTM standards. But that is not enough. Such important concepts require much more attention and the development of the wider range of educational activities. The “new mathematics” movement has given us a wealth of material from the field that we can, taught by history, easily transform into modern standards. For example, why stop at a comparison relation that is usually associated with some future acquiring of measurements (smaller – bigger, lighter – heavier, etc.) or an equivalence relation (same height, same shape, same color, etc.)? Why not use graphs to represent other relations? Graphs allow children to visually analyze the entire menagerie of relations from their world. Such a presentation of relations is very striking. Willy Servais (Servais, Varga 1971: 97) writes: “Arrow graphs are used to represent binary relations by sets of arrows […] The finished graph, being formed of arrows, preserves the memory of the dynamic operation involved in drawing it. […] They are really perceptual drawings fulfilling an abstract purpose. Colored graphs have made a powerful contribution to the elementary understanding of relational notions […].“ E.g., we can paste or draw the characters on paper and connect them with arrows: blue for “to be a mom of”, red for “to be a dad of”. In this graph, children can explore family relationships; for example, find all a person’s grandparents, or all her siblings, etc. Thereby, I think it’s important to represent people on graphs by pictures and not by names. In my opinion, *writing and reading should not be present in mathematical content at this level because children are not fluent in these: writing and reading add unnecessary burdens and bring additional abstraction that destroys the simplicity of basic mathematical content.* We must not take written content lightly into mathematical activities. The NCTM standards do not take care of that. Furthermore, just as we can expand the mathematical content associated with relations, we can also expand the mathematical content associated with sets and functions. E.g., we can introduce operations with sets, not directly but by merging language conditions using connectives “not”, “and” and “or”. Thus, we teach children the correct logic of language, as demonstrated by Zoltán Pál Dienes in a lesson in logic (Servais, Varga 1971: 38–46). The NCTM standards describe various activities with functions (matching, patterns, geometric transformations, symmetries, etc.), but why not add functions that are constantly present in the children’s world, such as “mom of” and “dad of”, which can be combined in interesting ways for children, for example, using the graphs described above? Or movements in space (forward, backward, left, right, etc.) which can also be combined in interesting ways, for example, to discover which composition of movements can undo two steps forward, turn right and three steps backward, or to discover different compositions of movements that lead to the same result (the final position and orientation of the body).

4.2. GEOMETRY

Next to these basic mathematical elements are the mathematical elements that arise from the child's movement, navigation, and construction in space. This includes distinguishing directions and rotations, along with the "amount" of movement in a direction or in rotation. With their development, the child establishes control in space. These activities are described in the NCTM standards, but I think they are far more important than learning geometric shapes which the NCTM standards give priority to. Of course, the figures are present in the surrounding area. But it is a space designed by adults. When we transfer these figures into children's space, we must be aware that these figures do not have the same importance in the children's world as in the adult world. My limited experience has shown that in the children's world, circles, triangles, rectangles, etc., are not as prominent as they are represented in the NCTM standards. For example, Nina uses them only in the construction of patterns that are interesting to her, or they are attractive to her because of their possible symmetry. But she doesn't really care how many sides a figure has, which figure has more sides, etc. She only learned to recognize a rhombus, just because that word was interesting to her. But she showed no interest in identifying which properties characterized the rhombus in relation to other figures. I can't imagine a motivation in the children's world that would lead to identifying and analyzing the properties of geometric figures. My thesis is that children simply use figures at the preschool level but do not analyze them, just as they use the predicate "to be blue" and do not analyze it. *Children's space is primarily a space of their movements, navigation in space, and constructions in space, and the development of these abilities should be emphasized in their geometric upbringing.* In developing these abilities today, physical education helps them far more than mathematics education standards.

What is still important about geometry at this level, and which in my opinion is not adequately represented in the NCTM standards, is that geometry provides great opportunities for visual representation of problems by which a child can create mathematical models of various situations. Ordinary drawing of an elephant, for example, is the creation of a mathematical model of an elephant. Here one can follow how the child creates an ever-better model of an elephant over time, even varying the model depending on what interests her in the elephant. We can draw a strong analogy of these children's models with the mathematical models used by adults. These children's models are the first steps in modeling increasingly complex situations. Not to mention that in this way children develop a sense of space and control of lines and shapes in space, especially if they model not on paper but with some material in space. A step forward is sketching the space in which a child lives, from a sketch of the room to a map of the entire area in which she moves, as well as sketching her movement in that space using straight or curved arrows. *Making spatial maps as well as using ready-made maps and solving various problems with* *the help of maps is very important for the development of the child's mathematical abilities* and should be given more importance and more attention in mathematics education. This is very well recognized in *The National Geographic Network of Alliances for Geographic Education* (National Geographic 2022).

4.3. NUMBERS

Numbers are the oldest and still the most important mathematics. However, in my opinion, *natural numbers are too much imposed on the children's world and as such overshadow other mathematical content – they can even turn children away from mathematics due to their more pronounced formal aspect*. That is why numbers should be treated more carefully with the youngest than is the case now. My suggestion for preschoolers is as follows. By comparing sets by establishing a 1–1 connection between their objects, children turn their intuition of quantities into a precise mathematical model of comparing sets (it is better not to mention sets) – where there are more, where there are fewer, and where there are equal objects. The next step is to introduce numbers and a counting process that establishes a 1–1 connection with the initial segment of the set of numbers, and thus the quantities are represented by numbers. *At this level, natural numbers for children are nothing but spoken words that have a certain order in speaking*. In the Croatian language we have a series of words: "jedan, dva, tri, …". When children in Croatia learn English, they easily replace Croatian numbers with isomorphic English numbers: a new set of spoken words: "one, two, three, …". It is important to emphasize that children's numbers are always concrete objects, spoken words, and not, for example, "equivalence classes of sets according to the relation of equipotency" as the creators of "new mathematics" tried to present them to children. Today it is often imposed on children that numbers are (represented by) written signs (numerals). In my opinion, such an approach is wrong for several reasons. First of all, numerals do not have the natural order that spoken words have in chronological order, which is crucial for the counting process.[8] Furthermore, they are symbols and as such introduce at this level unnecessary abstraction into the counting process. In addition, they require a certain child's reading and writing skills, which, as I pointed out above, is a complex process that unnecessarily burdens the mathematical content. By counting, children can easily compare sets of objects by comparing the associated numbers: which numbers occur first and which later in the number sequence. Addition and subtraction of small numbers at this level can be done by adding and subtracting sets of objects that they represent, but not directly by operating with numbers. Direct operations with numbers (apart from the operation of taking the next number)

[8] This is in line with Kant's well-known claim in the *Prolegomena* that arithmetic "forms its concepts of numbers through successive addition of units in time".

not only require that children know how to write and read numbers, but they are of a formal nature which in my opinion is not part of the children's world at that age.

4.4. A NOTE ON OTHER MATH ELEMENTS

There is a whole series of other mathematical elements that, in my opinion, need more attention than currently given in the standards, which I will not deal with in this article. These are, for example, simpler mathematical structures (they can be developed through games that do not have to be competitive games but also cooperative games), graphs (to represent spatial networks, relations, states and changes, etc.), recursion (basic elements plus construction rules), topology (dressing, knots, transformations in clay, stretching rubber, etc.), chance (games with an element of chance), change (dynamics of movement and activities), etc.

5. BACKGROUND MATHEMATICAL ELEMENTS

In addition to primary mathematical elements, attention should be paid to secondary mathematical elements, elements that are present in all mathematical activities. Some of the elements have already been mentioned above: these are sets, relations and functions that appear in the children's world as primary mathematical elements through concrete examples. Then, there are abstraction, representation, procedural activities (algorithms), logic and language. But in my opinion, language is the most important, so I will dwell on it, especially since it includes both abstraction and logic. Representation has already been mentioned in the context of geometric representation of problems.

5.1. LANGUAGE

Language elements are concrete means from our world of internal activities by which we control reality. Thus, language means form a very powerful mathematics. By choosing words in a situation, we do an abstraction, extracting from that situation what interests us and abstracting the rest. It is an essential mechanism that helps us deal with the complexity of the world. Furthermore, we use words to control and structure the aspect of the situation that interests us. Through noun expressions we control objects, through predicate expressions we control, and I would say we refine and create concepts[9]. Thus, *language itself is an important type of mathematics that should be developed at the preschool age as well.* Like us, a child

[9] In Čulina (2021) the key role of language in our rational cognition and thinking is described.

manages to control and understand reality through language. That is why we help her a lot in mathematical development whenever we read her stories, when we listen to her talk, and when we encourage her communication with other children and adults. Of course, this attention to language development should also be nurtured in the child's mathematical activities. I would like to mention once again that at that age, *language is the means of the child's activities and not the subject of his activities.* By helping a child to develop language in a given mathematical activity, we help her to learn abstraction and to clarify the concepts or meanings of words – to clarify her mathematical means. E.g., by pointing her to triangles and quadrilaterals in composing tangrams we help her to abstract irrelevant elements (color, type of material, …) and single out relevant elements (shape and dimension) to solve problems. We also help her to specify the concept of triangle, that at some point both equilateral and right triangles are triangles, and that a parallelogram is not a triangle. In short, *by refining the language, the child refines his mathematics*. Furthermore, *by using language, the child opens the way to the idealized mathematical world that arises from her activities, thus expanding her mathematics*. This step is not a problem for the child either. Just as she uses language to specify the story of Snow White and the Seven Dwarfs, she uses language to specify the world of "all numbers". The NCTM standards do not recognize language as a very powerful mathematics and as a means of building idealized mathematical worlds, but they do recognize the importance of language as a means of clarifying and communicating mathematical activities.

5.2. LOGIC

No matter how we look at logic, it always manifests as the logic of a language. Thus, *by acquiring a language, children also acquire logic*. I have already mentioned the use of connectives in classifying objects using complex conditions. My experience with Nina showed me that children learn the meaning of negation ("I'm not going to kindergarten!") and of conditionals (Me: "How can I help you stop your knee hurting?", Nina: "If I watch cartoons, it will stop my aching knee.") very quickly, and somewhat slower the meaning of conjunction and disjunction. Children also understand the meaning of quantifiers ("Macarena is always angry", "Is anyone here?"). Logical inference is not foreign to them, especially when it works in their favor (Grandma: "Santa Claus only brings gifts to good children", Nina: "Then Ezra will not get a gift", Grandma: "Why?", Nina: "Because he was not good: he hit me." – there are connectives and quantifiers in this conclusion). Furthermore, if there is inconsistency in the story, a child immediately registers it.

And consistency is the equivalent of logical reasoning[10] (Me: "What's your doll's name?", Nina: "Aurora", Me: "Wasn't her name Julia yesterday?", Nina: "Yes, but she's constantly changing her name."). Although the NCTM standards emphasize reasoning as a separate process in mathematical activities, the standards limit it to the process of establishing mathematical claims, and even in such a limited context, the view of children's reasoning is very limited. What is written in the NCTM standards on page 122 – "Two important elements of reasoning for students in the early grades are pattern-recognition and classification skill" – may be appropriate for chickens but certainly not for children who are full of imagination. The NCTM standards do not recognize children's thinking as separate mathematics that develops through all children's activities, especially through stories and fantasies, and not only in mathematical activities, nor do they recognize the overall richness of children's thinking. On the contrary, it is very important to encourage children to retell or invent stories and events themselves, to discuss stories and events with each other or with us, to look for reasons for certain behaviors or events, and to draw consequences from available information.

5.3. PROCEDURAL THINKING

Procedural thinking (*how to achieve something*) *is more appropriate to the dynamics of the children's world than declarative thinking* (*what is and what is not*). However, these procedures should be meaningful and expressed in spoken and pictorial language. The refinement of procedures should be gradual with the awareness that in this way freedom is lost but efficiency is gained. Finally, adults don't really like detailed instructions, but only general instructions that leave us a lot of space for our own creation. In my limited experience, this is even more present in children. *The transition to formal procedures, such as algorithms with numbers, is a demanding transition, because formal procedures involve writing, and they lose content, so they should not be rushed*. The NCTM standards deal only with formal procedures with numbers. As formal procedures are not appropriate for preschoolers, the procedural thinking of the youngest is not present at all in the NCTM standards. This omits one important mathematical component of the child development. It can be developed very efficiently through nursery rhymes, songs, spatial movement instructions, cooking recipes, etc. For example, with the help of "The Enormous Turnip" folktale, children learn the concept of iteration in problem solving (programming loops) and with the help of "Pošla koka na Pazar" (English translation: "When Hen Was on Her Way to the Fair"[11]) South Slavic

[10] In first-order logic, from a given set of assumptions a conclusion logically follows if and only if the set of assumptions together with the negation of the conclusion is inconsistent.

[11] I only know of the English translation in the book (Stanić 2018).

folktale, children learn the concept of reductive problem solving (subroutine calls in programming). The development of the procedural component in children is also important due to the increasing importance of software in modern society. If we leave out technology, programming is, from a conceptual point of view, part of mathematics. Praiseworthy is the emergence of simple programming languages and environments, such as Scratch (Scratch Foundation 2022), in which children can easily and vividly create characters, program their behavior, and compose stories. All this is an important part of mathematics for the youngest to which adequate attention should be paid.

5.4. PROBLEM SOLVING

And at the end, an essential component of mathematics is that it has a purpose: to be a tool of our rational cognition and rational activities in general. This is true for both adults and children. Only *the purpose of children's mathematical activities must be incorporated into their world.* Just as all human civilization has developed mathematics as a means of solving various big and small problems, and just as individuals are developing it, in the same way *children in their children's world need to build their mathematics by solving problems from their world. As in the world of adults, this purpose in the world of children gives mathematical activities integrity – a natural framework for their development.* This component, which is usually called "problem solving", must be kept in mind when helping a child to develop mathematical skills. This can be solving problems arising from the organization of the child's daily activities (placing goods in drawers), arising from play (how to assemble a crane from Lego bricks) or integrated into the world of a story (e.g., the story of the wolf, goat, and cabbage). Counting on its own can be fun, but it only gets real meaning when counting controls whether all the bears are present at the morning review of stuffed animals.

REFERENCES

Bilbao (2020): A. Bilbao, *Child's Brain Explained to Parents*, Plataforma.

Bourbaki (1950): N. Bourbaki, The architecture of mathematics, *The American Mathematical Monthly*, 57(4), 221–232.

Carrier et al. (1962): G. F. Carrier, R. Courant, P. Rosenbloom, C. N. Yang, H. J. Greenberg, Applied mathematics: What is needed in research and education: A symposium, *SIAM Review*, 4(4), 297–320.

Cantor (1883) G. Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, Teubner.

Čulina (2018): B. Čulina, An elementary system of axioms for Euclidean geometry based on symmetry principles, *Axiomathes*, 28(2), 155–180.

Čulina (2020): B. Čulina, *Mathematics – an imagined tool for rational cognition*, https://philpapers.org/archive/CULMA.pdf, unpublished.

Čulina (2021): B. Čulina, The language essence of rational cognition with some philosophical consequences, *Tesis (Lima)*, 14(19), 631–656.

Dedekind (1888): R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig: Vieweg.

English, Mulligan (2013): L. D. English, J. T. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning*, Dordrecht: Springer Science+Business Media.

Ernest (1997): P. Ernest, *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press.

Hersh (1997): R. Hersh, *What is Mathematics, Really?*, Oxford University Press.

Lascarides, Hinitz (2000): V. C. Lascarides, B. F. Hinitz, *History of Early Childhood Education*, Falmer Press.

Mac Lane (1986): S. Mac Lane, *Mathematics, Form and Function*, New York, Berlin, Heidelberg, and Tokyo: Springer-Verlag.

Moss et al. (2016): J. Moss, C. D. Bruce, J. Bobis, Young children's access to powerful mathematics ideas: A review of current challenges and new developments in the early years, In: L. D. English, D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education*, third ed., 153–190, Abingdon, New York: Routledge.

National Geographic (2022): National Geographic, *Spatial thinking about maps*, https://www.nationalgeographic.org/media/spatial-thinking-about-maps/, Accessed 2022/4/24.

NCTM (2000): NCTM, *Principles and Standards for School Mathematics*, The National Council of Teachers of Mathematics.

Phillips (2015): C. J. Phillips, *The New Math: A Political History*, Chicago: The University of Chicago Press.

Scratch Foundation (2022): Scratch Foundation, *Scratch*, https://scratch.mit.edu/, Accessed 2022/4/24.

Servais, Varga (1971): W. Servais, T. Varga (Eds.), *Teaching School Mathematics*, Middelsex, UK: Penguin Books – Unesco.

Smith (2001): S. S. Smith, *Early Childhood Mathematics* (second e.d), Needham Heights, MA: Allyn & Bacon.

Stanić Rašin (2018): I. Stanić Rašin, *When Hen Was on Her Way to Market: A Folktale-Inspired Story of Manners and Illustrated Nursery Rhyme*, Arlington, MA: Perlina Press.

Stone (1961): M. Stone, The revolution in mathematics, *The American Mathematical Monthly*, 68(8), 715–734.

Boris Čulina

Veleučilište Velika Gorica
Departman za matematiku

ŠTA JE MATEMATIKA ZA NAJMLAĐE?
(Šta je stari matematičar naučio o matematici od svoje unuke Nine)

Rezime: Prema filozofiji matematike opisanoj u radu, matematika je proces i rezultat oblikovanja intuicije i ideja o našem internom svetu aktivnosti u misaone modele koji nam omogućuju da bolje razumemo i kontroliramo celi svet. Pod „internim svetom aktivnosti" podrazumevam svet koji se sastoji od naših aktivnosti nad kojima imamo izrazitu kontrolu i koje organiziramo po vlastitoj meri (npr. pokreti u sigurnom prostoru, grupiranje i raspoređivanje malih objekata, prostorne konstrukcije i dekonstrukcije s malim objektima, govor, pisanje i crtanje po papiru, oblikovanje i transformisanje manipulativnog materijala, slikanje, pevanje i sl.). Iz tih konkretnih aktivnosti nastaju idealizirani matematički svetovi (modeli, teorije) koji proširuju i nadopunjuju interni svet aktivnosti.

Iz takvog gledanja na matematiku proizlazi i odgovor na pitanje „Koju matematiku treba da uče predškolska deca?”. Kao što su interne aktivnosti odraslih izvor matematike odraslih, tako su i interne aktivnosti dece izvor dečje matematike. One se najizrazitije ispoljavaju i najbolje razvijaju u dečjoj igri – štoviše, one su sama osnova dečje igre. Često je svrha dečje igre razumevanje vanjskog sveta (npr. „Igrajmo se doktora”). Kad se takva svrha doda igri, imamo u dečjem svetu matematički model istraživanog fenomena. Pouka je jasna: što je više igre, to je više matematike u dečjem svetu. Pored igre, deca razvijaju matematičke sposobnosti kadgod pokušavaju organizirati svakodnevni život uz pomoć odraslih (npr. rasporediti svoju robu po ladicama). Tako se dečja matematika sastoji od sveta dečjih aktivnosti koju oni eventualno svrhovito organiziraju da bi razumeli i kontrolirali vanjski svet i organizirali svoje delovanje u njemu. Ovakvo gledanje je posve u skladu s ustaljenom metodologijom obrazovanja po kojoj matematičke aktivnosti deteta moraju biti deo njegovog sveta: imati motivaciju, značenje i vrednost u dečjem svetu, a ne izvana, u svetu odraslih. Ukratko, dečja matematika je deo dečjeg sveta a ne van njega, i detetu pomažemo da razvija matematičke sposobnosti u kontekstu njegovog sveta a ne van njega.

U odnosu na ovakvo gledanje na dečju matematiku, uspostavljeni standardi matematičkog obrazovanja dece su preuski: niti pokrivaju sve značajne matematičke aktivnosti niti ispravno raspoređuju pažnju među aktivnostima koje pokrivaju. Previše se pažnje posvećuje brojevima, svi drugi matematički sadržaji se podređuju brojevima, dok je u geometriji previše pažnje dato geometrijskim likovima i telima, koji više pripadaju svetu odraslih nego dečjem svetu. U članku su opisani matematički elementi koje bi bilo poželjno da deca razvijaju, a kojima u standardima nije dana dovoljna pažnja ili nisu ispravno obrađeni. To su a) skupovi, relacije i funkcije, b) kretanje, navigacija i konstrukcije u prostoru, c) vizuelna reprezentacija problema, pogotovo pravljenje prostornih mapa, d) jezik, e) logika i f) proceduralno razmišljanje.

Ključne reči: matematika za predškolce, standardi matematike za predškolce, NCTM standardi, pokret „nove matematike”.

Ivana M. Milić
University of Kragujevac
Faculty of Education in Jagodina
Department of Didactics and Methodology
Jelena M. Mladenović
Department of Mathematics and Informatics

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FROM PLANT MORPHOLOGY TO RHYTHMIC PATTERNS (OF MUSIC): A STEAM APPROACH TO STUDYING RELATIONS IN MATHEMATICS[1]

Abstract: In this paper, innovative procedures in working with students, applying the STEAM approach, and the possibilities of improving the quality of university education are reviewed from a theoretical perspective. The modernization of university teaching implies following global trends, with the primary goal being the formation of versatile, competent students who will be able to respond to the modern demands of society and participate in the exchange of information related to current scientific achievements while constantly strengthening their own capacities. During university education, it is necessary for students to develop their potential and competencies, have positive attitudes towards their future in the educational profession, and understand the importance of their teacher role when choosing an approach to educational work in kindergarten.

The competence of the preschool teacher, as well as students, is reflected in the implementation of activities, activation of children's potential, enrichment of children's experiences, encouragement of creativity, and close exposure to phenomena and processes in the fields of science and art. Therefore, it is important to familiarize students with innovative approaches to educational work in kindergarten because they will be able to transform and properly apply the acquired knowledge later in their future work in order to ensure the holistic development of children.

The paper presents an example of an activity that integrates the contents of the subjects of mathematics, natural sciences, and music, with the aim of highlighting the possibilities of improving university teaching through different approaches. Presented examples can be applied during the education of students (future teachers), which would strengthen their competences for working with children of preschool age.

Keywords: rhythm, botany, mathematical relations, STEAM approach, students preschool teachers.

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ivana.milic75@gmail.com

INTRODUCTION

The modernization of university teaching implies following global trends, where the primary goal is the formation of versatile, competent students who will be able to respond to modern demands and participate in the exchange of information about scientific achievements, while constantly strengthening their capacities. The expected outcomes of the modernization of university teaching are that students become aware of their potential and competencies, recognize the importance of the teacher's role in choosing an approach to educational work in kindergarten, and develop a positive attitude toward their future in the educational profession. They must understand that, according to the *Foundations of the Preschool Education Program* (*Godine uzleta*), the profession of a teacher is a unique profession compared to all others; it is ethical and reflexive in its essence and is based on complex and unique competencies.

The teacher is the one who directs the process of learning and enriches the experiences of children in kindergarten. Children primarily learn from their own experiences, interactions with other children, and conversation, through which they come to conclusions. For this reason, the primary role of the teacher is to support the children in the process of acquiring knowledge and experience by creating a stimulating environment. At the same time, they will develop their personal competencies over time, such as evaluation of personal practice, observation, and documentation of children's activities and knowledge of scientific concepts.

Therefore, it is important for students to be familiar with innovative approaches to educational work in kindergarten. There is a need for examples of good practice, so that they could transform and properly apply the acquired knowledge in their future professional work.

Nowadays, the need to integrate content from different fields – including the natural sciences, social sciences, and even art – and to observe and understand the essential connections and relationships between them, is increasingly being adopted. We found the initial connection in the theoretical relations of mathematics, natural sciences, and music. According to Despić (1997), "mathematical laws in music allow us to describe the metrics and development of a musical piece in precise language, including the passage of time in it". Other examples that indicate these connections can be found; the time signature in music is represented by a fraction, as in mathematics. The principle of dividing a whole into parts in mathematics is also applied in music if the even division of the whole note is observed (Despić 1997). Rajić, in his paper, concludes that "just reading durations and ratios regarding the duration of notes requires basic knowledge of mathematical fractions. In this way, the connection between mathematics and music is highlighted once again" (Rajić 2019: 80).

Also, in describing natural phenomena and processes, even the characteristics of living beings, the use of mathematical language is a necessity.

THE STEAM APPROACH IN EDUCATION

In recent years, modern educational practices have seen the integration of two terms that were previously considered separate. According to scientists, the established term STEM, an acronym for science, technology, engineering, and mathematics education, should have been expanded. STEM education emphasized theoretical understanding of solutions to real problems. Although invisible at first glance, art has always played an important role in STEM education. That's when STEAM, a term that included art, was born (Swe Khine, Areepattamannil 2019).

Scientific inquiry as a method of scientific research conveys claims based on facts. This research, like art, is based on creativity and curiosity. For solving scientific, but also technological, engineering and mathematical issues, flexibility in choosing the appropriate methodology is of key importance. Thereby, it is necessary to establish the roles played by the teacher and the student so that the research goes in the right direction. A teacher should provide direction that points to a solution, but not offer a solution. The teacher is there to teach the student how to learn. The student should have an active role in the learning process. Unlike traditional methods, inquiry-based instruction requires full student engagement. When designing a problem-solving method, it is important to consider the level at which the student can take on such a role and gradually guide the research process (Swe Khine, Areepattamannil 2019).

It is often common for students to find STEM classes boring. It can be transformed by implementing the arts (STEAM). Also, the advantage of introducing art subjects into the educational process is the balanced emotional, psychological, and intellectual development of each individual, as well as society as a whole (UNESCO 2012).

STEAM is an evolving educational model that demonstrates how traditional academic subjects such as science, technology, engineering, art, and mathematics can be structured into an integrative curriculum planning framework. As a pedagogical framework, STEAM often includes educational frameworks and practices in which a set of disciplines is considered the core of the learning experience or is seen as the primary and only subject that uses another discipline to achieve its goals (Mejias et al. 2021). STEAM fields have multidisciplinary, interdisciplinary, and transdisciplinary approaches, and are suitable for achieving learning outcomes. Such an approach includes general development as well as development specific to an individual discipline and is related to integrative or holistic education. Research on these and similar educational relationships of individual disciplines (science and art) is currently present in the world as a way to find common education goals (Yakman 2010). There is a need to connect the individual discipline with others in a structure that can accommodate many combinations of disciplines.

However, there is controversy among experts dealing with this approach as to how it should be implemented in the education system. On the one hand, there are those experts who recommend that this approach be developed in a transversal way from all areas of the curriculum, enabling teachers to present integrated lessons. That way, students would learn while doing (working on the problem). On the other hand, there are opinions that it is impossible for one person to teach STEAM transversally through the subject because she or he doesn't possess knowledge from other areas of the curriculum. There are also efforts to establish an area within the curriculum to ensure a common methodological line between subjects (Duo-Terron et al. 2022).

With the help of modern technologies, the range of teaching aids and materials used has been significantly expanded. Several techniques can be used in STEAM education to improve pedagogical effectiveness, encourage scientific thinking, and raise the appreciation of science. Some of them involve multisensory creativity in different environments and active participation of individuals (children, students). STEAM is an increasingly popular pedagogical approach to enhancing students' creativity, problem-solving skills, and interest in individual STEAM fields. It emerged in response to the need to increase students' interests and skills in science, technology, engineering, and mathematics (Perignat, Katz-Buonincontro 2019).

With the STEAM approach, opportunities are opened for students to develop competencies by mastering morphology, as well as phases in setting the rhythm, and educational procedures for developing concepts about certain spatial dimensions.

In the example activity presented in this study, students need to recognize the plant species and be able to describe its morphological characteristics, then structurally connect it with the rhythmic patterns and the performance of different note durations (certain rhythmic figures). Only then the student will be able to successfully transform his knowledge to form the notion of size relations (in this example, *big–small*) in children. This is the key element that characterizes the STEAM approach because natural and mathematical contents are not studied individually, although certain discipline-specific knowledge is necessary, but integrated with music.

Students are expected to find contents within three different subjects that are suitable for connection and to transform and present them so that the knowledge of plant leaves morphology (botanical aspect) is placed in relation to size (mathematical aspect) and forms rhythmic patterns (musical aspect). In this way, rhythmic patterns are represented by rhythmic images, which allows students to transform complex phenomena into concepts that are close and comprehensible to preschool children. Further expert guidance on this "transformation of image into sound for following the rhythmic flow when reading musical notation" (Vasiljević 2006: 199) with notes, which children acquire in the field of musical literacy only in elementary school, will contribute to children mastering music verticals in the period of conscious musical literacy, and facilitate the performance of the written rhythm.

As in the native language, we translate visual symbols into sounds, and by looking at letters and connecting them into words, we move on to reading. Therefore, by looking at the shapes of leaves and determining the relations between them (without naming the spatial dimensions, but by directly transforming the acquired knowledge), we audibly perform rhythmic patterns.

More specifically, from the aspect of teaching rhythm, the STEAM approach in this way introduces students to one of the most complex pedagogical tasks in the field of rhythmic reading: *parlato*. The students must have the ability to read and keep a regular rhythm pulse, move their gaze forward along the rhythmic pattern while maintaining the tempo, and develop the skill of "following the musical text along the difficult wide horizontal line system" (Vasiljević 2006: 200) in all forms of music performance in their professional work. In this way, they will be able to identify the same abilities in children later in their work.

THEORETICAL CONCEPT OF RHYTHM

The basic expressive elements of music are melody, harmony, and rhythm. In music theory, rhythm is the alternation of notes, rests, and silences in time. Rhythm consists of sound, silence, and accents in music. When a series of notes and rests repeats, it forms a rhythmic pattern. Musical rhythm also determines how long notes are played and with what intensity. This creates different note durations and different types of accents. Rhythm allows the music to move forward, animates the piece of music, gives structure to the composition, and affects the character of the music. Most classical musical ensembles include percussionists, the so-called *Rhythm Section*, who maintain the rhythmic backbone of the ensemble as a whole, regardless of the fact that all members of the musical ensemble bear equal responsibility for their own rhythmic performance, the performance of musical measures, and the rhythmic patterns indicated by the composer of the piece of music.

In order for children and students to be able to identify rhythmic durations, it is expected that they possess a certain level of rhythmic abilities. It is necessary for the teacher to know the elements of rhythmic abilities and then to state a child's developmental stage of certain rhythmic abilities at a specified age.

It is important that students, through the process of university education, learns which elements of rhythmic abilities they could and should foster in preschool children. That is the starting point.

In general, from the perspective of rhythm methodology, we represented which elements of rhythmic abilities can be taught:

- Ability to remember tempo (the perceived frequency of musical pulse with a perceived pulse or beat);
- Ability to adapt to a given tempo and correspond to changes in tempo;

• Ability to perceive and perform different rhythmic types;
• Ability to recognize agogic nuances of rhythm;
• Ability to polyphonically follow different rhythmic relations between voices when the musical lines are rhythmically differentiated (Vasiljević 2006).

Furthermore, with the correct methodical solutions, it is possible to nurture rhythmical pulse and grouping into units through moving in a circle, marching, performing, dancing, and singing songs with children of preschool age.

However, the ability to adapt to a given tempo and correspond to changes in tempo, as well as the ability to perceive and perform different rhythmical types, is not possible to develop without a sufficient professional teacher. Meanwhile, the ability to polyphonically follow different rhythmic relations between voices when the musical lines are rhythmically differentiated could be developed only through education in a music school. Therefore, at preschool age, the teacher can monitor and influence the development of only certain rhythmic abilities.

Finaly, professional guidance is necessary for improving and consciously developing skills for keeping an equal pulse, maintaining rhythm, and developing a sense of dynamics until children begin school.

A positive result from such directed development and accumulated unconscious reception of musical influences and sound layers (Vasiljević 2006) will not be absent. Continuous and spontaneous experience and performance of different rhythmic patterns in the phase before musical literacy is of great importance for children's later awareness of certain phenomena (Plavša, Popović, Erić 1961).

Therefore, through university teaching, it is necessary to direct students' activities towards personal and professional development.

BOTANICAL ASPECT

According to the program for the education of preschool children, one of the goals is for children to be familiar with the living world that surrounds them. To be able to achieve such a goal, they need to learn to recognize characteristic species of plants or animals according to certain rules, learn to notice details specific to individual species, and compare important characteristics to be able to conclude what species it is.

It is clear that, above all, children should have a competent teacher who can teach them that. For this reason, one of the goals in the curricula of subjects dealing with these topics is to enable students, and future teachers, to recognize certain species from the immediate environment. In this sense, students should know specific plant species and have the competence to transfer this knowledge to children of preschool age.

Knowledge of plant morphology, the science that studies the external appearance of plant species, i.e. the appearance of plant organs such as leaves, is useful in describing plant species. Although the external appearance and size of the leaves of different plant species is not the only characteristic based on which the species can be identified, describing their appearance, observing details, and noticing similarities and differences in the appearance of the leaves is a good starting point for studying (and describing) the plant world and the environment.

When describing plant species, it is necessary to observe as many details as possible that characterize the given species. Such details can sometimes go unnoticed, so students should be trained to spot and identify those details and then direct the dialogue with the children properly. So, for example, some species differ among others in the size of the leaf or in the shape of the leaf, or the way the leaves are attached to the stem. When describing, quantitative properties are used, which in mathematical language belong to size relations – *height*, *length*, *thickness*, and so on (Egerić 2006; Najdanović 2012). Children notice such relations by comparing objects of the same shape, or in this case, plant leaves. Based on this, they adopt concepts that are opposite in meaning: *big–small*, *high–low*, *long–short*, *wide–narrow*, *thick–thin*, and *deep–shallow* (Egerić 2006; Najdanović 2012). The relations *up*, *down*, *in front of*, *above* and so on are also described, which belong to positional relations. After noting the details based on the observed characteristics, the given species is classified into a certain category of affiliation (taxonomic category) and identified as a specific species. Mathematically, this would correspond to an inclusion or subset relation.

Familiarity with the plant world begins at an early age during preschool education and continues during further schooling. Numerous teaching aids are available today, such as botanical atlases, laboratory manuals textbooks, natural or digital herbariums, as well as live plant material help children and students get to know the diverse plant world. For children to be able to distinguish one species from another when describing the essential (key) characteristics of species and comparing those characteristics, they need to be familiar with certain mathematical relations such as position relations, size relations, or inclusion relations.

To present the STEAM approach to students and show through a practical example how to apply it in work with preschool children, we chose a plant: the plantain. Plantains inhabit children's immediate environment in meadows, city parks, and lawns, so they are familiar with it visually, but they cannot identify and distinguish it from other similar species by certain key characteristics.

In the flora of Serbia, several plant species which bear the common, folk name plantain are known. These are mostly herbaceous perennials, less often bushy plants from the flowering plant's clade (Magnoliophyta), with a cosmopolitan distribution. Most often, they can be found in the composition of plant communities of meadows and pastures in lowlands, mountain grasslands, and subalpine shrublands. The plantain genus (Plantago) has over 200 species (Tabašević et al. 2021).

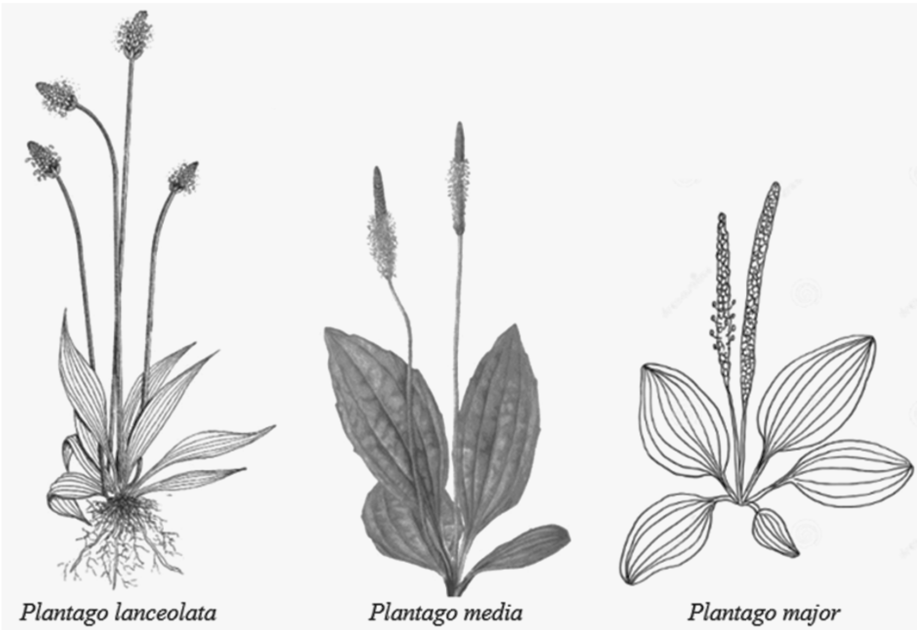
What most species of this genus have in common is the position of the leaves: an alternate arrangement at the base of unbranched stems in the form of a ground rosette. Because of that characteristic way the leaves of some lie flat on the ground, this genus bears the scientific name Plantago, derived from the Latin word for the sole (foot-sole-like, feminine termination of planta, ancient Latin, plantaginem) (Gledhill 2008).

Considering that it is difficult to determine the phylogenetic affiliation within the genus as well as in higher taxonomic categories based on morphological characteristics alone, modern science applies analyses based on DNA sequences and chemotaxonomic research. This is because the species of this genus are characterized by specific chemical compounds and in folk medicine, they are known as plants with medicinal properties. If we exclude those analyzes and get to know the plant world during the education of preschool children, individual species of this genus can be distinguished by the shape and size of the leaves. The shape, size, and structure of leaves vary considerably from species to species of plant, depending largely on their adaptation to climate and available light, as well as other ecological factors (Janković, Gajić 1974).

Plantago lanceolata, *Plantago media*, and *Plantago major* can be found in our meadow ecosystems (Picture 1). *Plantago lanceolata* is known by several common names: narrow-leaf plantain, ribwort plantain, lamb's tongue, buckhorn, and, in Serbia, male plantain. *Plantago media* is known as the hoary or medium plantain, while *Plantago major* is known as the broad-leaf plantain, white man's footprint, greater plantain, or, in Serbia, female plantain.

The narrow-leaf plantain (*Plantago lanceolata*) has elliptic to lanceolate-shaped leaves, pointed at the apex, with a smooth margin. The basal leaves are lanceolate spreading or erect, scarcely toothed with 3–5 strong parallel veins narrowed to a short petiole. The medium plantain (*Plantago media*) has finely-haired leaves that are broad, elliptic, or ovate in shape, usually twice as long as they are wide, sessile, apetiolate (without a leaf stalk) or at the base narrowed into a short and wide petiole. In the broad-leaf plantain (*Plantago major*), the leaves are broadly ovate to elliptic in shape, with an acute or blunt apex and round base, with a smooth margin or toothed in the lower part and distinct petiole almost as long as the leaf itself or longer. There are five to nine conspicuous veins over the length of the leaf (Mišić, Lakušić 1990). Due to the shape and size of the leaves, which are larger than those of other Plantago species, it was given the scientific name *major*.

Picture 1. Illustrations of different Plantago species



THE PROCESS OF ACQUIRING KNOWLEDGE ABOUT RELATIONS OF SIZE

Because children of preschool age have great potential to form elementary mathematical concepts and raise them to a higher level (here we distinguish the spatial dimensions of the subject), that period should be used in the best way. In mathematics, the cognitive process takes place through experiences and the senses, in which two phases can be distinguished: the perceptual phase and the phase of thought processing, in which the idea of a concept is created. Therefore, the formation of a mathematical notion is a process of knowing where a sensory experience is invoked in thought processing, a reminiscence of memory that children already have (Egerić 2006).

In that regard, it is necessary to choose the correct methods and various work approaches to gradually, through play and fun, influence and mathematize the appropriate notions of relations by noticing and emphasizing important mathematical features (Najdanović 2012).

In mathematics, relationships and connections between elements represent relations. From the earliest age, in everyday life situations, children are exposed to and surrounded by those relations. Through the activities organized by the teacher, children through play can observe the important characteristics of objects that are in a certain relationship. To describe their observations, they use terms from everyday speech close to them. Through the educational process, those terms will become notations and symbols for appropriate relations. With the professional guidance of the teacher, children will gradually adopt the terminology of mathematical notions and build clear ideas about their meaning.

When developing initial mathematical notions, children's activity, initiative, and communicativeness are of particular importance. To determine the spatial dimension (where objects are located) and to express their differences in size (*big* and *small*), children spontaneously come into contact with relations. At preschool age, children have certain ideas about relationships in their spatial environment. They already, in everyday communication, use sentences in which they express these relations; for example, size relations: the cat is *bigger* than the kitten; the chicken is *smaller* than the hen; Emilia's apple is *bigger* than Sofia's; Dimitrije has a *smaller* flower than Xenia; Vasa and Mata have balls of the *same size*, etc.

Those specific examples, which are the object of children's interest or represent situations from their lives, should be used and expertly transformed into a conscious understanding of relations. It is necessary to stimulate the thinking activity of children through well-organized play and a proper selection of didactic materials so that familiarization with the notion of relations flows from the concrete to the abstract. This is the primary task of the teacher, who is expected to carefully formulate questions that will guide children to find answers and solutions on their own. With this kind of organization, children are not deprived of the beauty and pleasure of discovery. Therefore, it is necessary to support students, future preschool teachers, and offer them the best possible solutions for managing activities while following the children's interests.

In a practical example, we chose plant leaves from the children's immediate environment (the leaf of the plantain) and placed them in mathematical relations, so that the children understood the concept of rhythm. We aspire to encourage students to innovate their approaches in future work by applying previously acquired knowledge.

PRACTICAL EXAMPLE – FROM PLANT MORPHOLOGY AND RHYTHMIC DURATIONS TO THE SIZE RELATIONS

The results of study in that field (Blatnik 1988) have shown positive effects of the visual perception field on children's cognitive processes when solving problems, more effective learning and understanding of content, and greater motivation and level of critical thinking. In general, the formation of notions requires the conscious engagement (activity) of children, because they acquaint with the world around them through their senses, practical actions, and mental operations. Here it is important to point out that children have already been exposed to certain (musical, mathematical, biological) experiences, and from a musical point of view, they possess certain rhythmic, perceptual, and reproductive abilities.

The senses of hearing and sight are considered by certain authors to be more perfect, superior senses because they perceive color and three dimensions (lines, surfaces, space) and have a common role in the auditory and grafic notation of tone (Vasiljević 2006). Because the visual field dominates human communication, by visualizing concrete content in this way, we can directly indicate the essence of the topic or unit that is articulated in the work plan. For “non-musicians” in the process of auditory perception of the musical flow, “concretization” in the form of a visual is necessary (Vujošević 2017).

Rhythmic images within the music-pedagogical practice represent a surface-spatial system, in which a certain melodic or rhythmic motion can be represented by visual symbols (Plavša 1989).

In the given example, we indicate a synergy action of the auditory and visual through the STEAM approach creates the possibility of multi-layered perception, not only of the rhythmic flow but also the perception and understanding of the contents of the other two areas. In this particular case, it will be presented to students as a way to successfully structurally connect the contents of three different subject areas: Development of initial mathematical concepts, Musical preschool education, and Kindergarten Natural Sciences.

Acquiring knowledge about the mentioned notions begins with observing “pictures, drawings, models”, directing children to “manipulate” them and to perceive common features while keeping them in their minds. “Thinking operations that process sensory experiences in the cognitive process of a concept” (Egerić 2006: 18) are analysis, comparison, synthesis, abstraction, identification, and generalization.

Children’s mental operations, in the given example, should be focused on all three areas using the STEAM approach.

Children observe the presented plants, and we expect them to be able to recognize and name them (classify them based on morphological characteristics), describe what the leaf of a given plant looks like, and even illustrate it. They then connect those visual representations with the auditory perception of longer and shorter rhythmic durations (musical aspect), compare them, and notice differences and similarities. It is necessary to explain to the students that, with direct questions, they should make the children perceive and connect the observed properties into meaningful wholes, from all three aspects. After that, with their expert guidance, direct the children’s thought operations to single out only the essential properties, which are, from the mathematical aspect, quantitative relations and spatial shapes. The process should flow in the direction that the properties of the material nature, which are concrete, become abstract (Dejić, Egerić 2006). Observed mathematical relations of size, by mental transmission, should be perceived as grafic notes and rhythmic durations, which represent preconceived rhythmic patterns.

By encouraging children to perceive certain quantitative properties in an organized manner, we form the notion of size relations. Thus, in Example 1, by comparing plantain leaves, which are of the same shape, but different sizes, and therefore from the mathematical aspect opposite in meaning such as *big* and *small* (or in some other examples they can be *high–low*, *long–short*, *wide–narrow*, *thick–thin*, *deep–shallow*, *heavy–light*), and from the musical aspect they are rhythmic images that represent rhythmic patterns (eighths and quarter notes), the notions of all three subject areas will be adopted.

Given that the terms *big–small* have a relative meaning (Egerić 2006) and that we do not tie the selected rhythmic images to one constant symbol but form a system of symbols according to each specific situation (Plavša 1989), we find a space to represent them through the plant leaves. The application of adequate illustrations, visual representation when adopting certain notions, and application of technology (dynamic mathematical software) provide better opportunities in designing visual and dynamic models in work (Milikić, Vulović, Mihajlović 2020).

It is necessary to establish and perceive relations during the auditory perception of longer and shorter note durations and relations between plant leaves when visually perceiving the appearance of a plantain leaf, and then distinguish and name *big* and *small*, which means understanding the elementary concept of the size relation. Students are expected to pose a problem to the children, which they will be able to solve only gradually, in stages, and connect all three subject areas through the STEAM approach.

Picture 2. Example 1. Rhythmic images represented by a plantain leaf in big–small relations

Rhythmic images will present rhythmic patterns, placed along an imaginary horizontal line on the surface or in space. For the auditory performance reading the rhythmic patterns, the student can choose the syllables of the plant name.

Rhythmic durations are represented by plantain leaves in two different sizes, in an approximate ratio of 1 : 2. A quarter note is a longer note duration and is represented by a big rhythmic image, a big leaf, and an eighth note is a shorter note duration and is represented by a small rhythmic image, a small leaf. In further work in the field of rhythm, we can include body movement, where rhythmic images would be shown by hand, and associatively enforce precisely defined rhythmic durations to be performed.

Students can pre-design a rhythmic notation as in Example 2.

Picture 3. Example 2. Rhythmic Notation

Moreover, we can set an additional creative task for the students: to create a literary text by themselves for the mentioned rhythmic patterns or to choose a thematically appropriate counting-out rhyme, thus introducing another subject area (speech development) as was done in Example 3. The metric of a literary text (sylabical durations) directly determines rhythmic durations and facilitates the notation of rhythmic images.

Picture 4. Example 3. Counting-out rhyme “Bokvica”

The main goal of using music and mathematics together is to use the power of music to engage children to make mathematical relationships by the use of music stimulus. We must begin by developing an activity that facilitates the construction of mathematical knowledge by encouraging the children to think mathematically and then add musical elements to enhance the activity (Mazzocco, Feigenson, Halberda 2011).

In the Example 4 children should create numerical patterns with flowers and place as many flowers as the number of claps they hear (Picture 5). We performed lyrics of the song named “Visibaba” (engl. Snowdrop – lat. *Galanthus nivalis*) and children have the task of listening to rhymes and a beat (pulse) in music which we clapped and place the specific number of Snowdrop flowers as they hear in the first vase. There are eight vases – as many as there are measures in the song. Every vase represents one measure and rhythmical pattern. Furthermore, in this example, children form mathematical sets by grouping Snowdrop flowers in vases and by visualizing the object. At the end, after the task has been solved, we could ask children to count flowers. In the first vase/measure children count to four, because there are four eighth notes, in the second children count to two, because there are two quarter notes and etc. In this case, a motivational musical environment is vitally important and can enhance future abilities in mathematics. We could provide new tasks for children such as counting numbers or grouping and comparing the groups of flowers by noticing the quantitative difference between the groups (more/less), or the quantitative ratio between flowers sorted out by the height criterion (tall/short). In every task it is very important to use correct mathematical language. We can also perform certain rhythmical patterns on some of Orf's instruments (wooden claves, maracas, wooden blocks) instead of clapping.

Picture 5. Example 4. Song "Visibaba"

Furthermore, in some other new example, we could show how students would practice the children's skill of counting by performing appropriate music games or rhymes and songs that mention numbers. Every practical example that we presented helped students strengthen their competencies for professional work.

CONCLUDING REMARKS

In accordance with modern tendencies in University education, it is necessary to use the potential of various approaches. Each approach may allow students to see their achievements and contributes to the improvement of their teaching practice (Semoz 2020). Although not all aspects of the STEAM approach have been seen in practice yet, we can point out that it contributes to the quality of students' knowledge, encouraging creativity and holistic education in general.

• Contemporary aspirations in early and preschool education pose many challenges to teachers that must be answered with decisive steps in innovations.

• It is possible and necessary to raise to a higher level the competencies that the future teacher should possess and develop through lifelong learning. Among others, these are to be dedicated to working, engaging, and finding creative solutions to problems. Furthermore, to be a collaborator, organizer, innovator, researcher, lover of music and science, and initiator of all projects to ensure the quality of the pedagogical climate, because their expertise and creativity influence the holistic development of each individual in the group (Milić 2016).

• Guidelines need to be given, as well as methods of work organization, didactic aids and materials, and examples of potential solutions that can support the student for the application of the STEAM approach in future work in pursuit of the acquisition of STEAM skills. The whole width of the space for possible new examples of STEAM activities that can be implemented with children in kindergartens (with clearly identified learning outcomes) should be perceived. Difficulties in implementing this approach can be overcome with professional guidance.

Our goal was not only to promote the methods and procedures of the STEAM approach but to highlight that university teaching is an open concept, whose processes inevitably require constant improvement and quality growth through access to different approaches.

REFERENCES

Blatnik (1988): F. Blatnik, Study confirms teaching with videodisc beats textbook methods, *T.H.E. Journal*, 16, 58–60.

Geist, Geist, Kuznik (2012): K. Geist, E. A. Geist, K. Kuznik, The patterns of music: Young children learning mathematics through beat, rhythm, and melody, *Young Children*, 67(1), 74–79.

Gledhill (2008): D. Gledhill, *The Names of Plants*, UK: Cambridge University Press, 4th Ed.

Dejić, Egerić (2006): М. Дејић, М. Егерић, *Методика наставе математике*, Јагодина: Учитељски факултет.

Dejić (2016): М. Дејић, *Предшколац у свету математике*, Београд: Креативни центар.

Despić (1997): Д. Деспић, *Теорија музике*, Београд: Завод за уџбенике и наставна средства.

Duo-Terron et al. (2022): P. Duo-Terron, F. J. Hinojo-Lucena, A. J. Moreno-Guerrero, J. A. López-Núñez, STEAM in Primary Education. Impact on Linguistic and Mathematical Competences in a Disadvantaged Context, *Frontiers in Education*, 7, doi: 10.3389/feduc.2022.792656

Egerić (2006): М. Егерић, *Методика развијања почетних математичких појмова*, Јагодина: Универзитет у Крагујевцу, Учитељски факултет.

Janković, Gajić (1974): M. Janković, M. Gajić, *Plantaginaceae Lindl*, In: M. Josifović, (ed.), *Flora SR Srbije,* 6, Beograd: Srpska akademija nauka i umetnosti, 318–335.

Kopas-Vukašinović (2017): Е. Копас-Вукашиновић, Могућности иновирања наставног процеса у систему универзитетског образовања, *Узданица*, XIV/1, Јагодина: Универзитет у Крагујевцу, Факултет педагошких наука у Јагодини, 7–15.

Основе програма предшколског васпитања и образовања – Године узлета, Retrieved in September 2022 from https://www.mpn.gov.rs/wp-content/uploads/2018/09/OSNOVE-PROGRAMA-.pdf.

Mazzocco, Feigenson, Halberda (2011): M. M. M. Mazzocco, L. Feigenson, J. Halberda, Preschoolers' Precision of the Approximate Number System Predicts Later School Mathematics Performance, *PLoS ONE*, 6(9): e23749. http://dx.doi.org/10.1371/ journal.pone.0023749.

Mejias et al. (2021): S. Mejias, N. Thompson, R. M. Sedas, M. Rosin, E. Soep, K. Peppler, J. Roche, J. Wong, M. Hurley, P. Bell, B. Bevan, The trouble with STEAM and why we use it anyway, *Science Education*, 105(2), 209–231, DOI: 10.1002/sce.21605.

Milić (2016): И. Милић, Ритмичке слике у функцији развоја предшколског детета, *Зборник радова Савремено предшколско васпитање и образовање: Изазови и дилеме*, књига 20, Јагодина: Посебна издања Факултета педагошких наука Универзитета у Крагујевцу, 231–244.

Milikić, Vulović, Mihajlović (2020): М. Миликић, Н. Вуловић, А. Михајловић, Геометријска интерпретација разломака применом образовног софтвера, *Узданица*, XVII/1, Јагодина: Универзитет у Крагујевцу Факултет педагошких наука у Јагодини, 307–317.

Mišić, Lakušić (1990): Lj. Mišić, R. Lakušić, *Livadske biljke*, Sarajevo: IP „Svjetlost", Zavod za udžbenike i nastavna sredstva; Beograd: Zavod za udžbenike i nastavna sredstva.

Najdanović (2012): M. Najdanović, Matematizacija relacija kod predškolske dece, *Sinteze*, 2, Kruševac: Akademija vaspitačko medicinskih strukovnih studija, 61–72.

Perignat, Katz-Buonincontro (2019): E. Perignat, J. Katz-Buonincontro, STEAM in practice and research: An integrative literature review, *Thinking Skills and Creativity*, 31, 31–43, doi: 10.1016/j.tsc.2018.10.002.

Plavša, Popović, Erić (1961): Д. Плавша, Б. Поповић, Д. Ерић, *Музика у школи I део*, Београд: Завод за издавање уџбеника Народне Републике Србије.

Plavša (1989): Д. Плавша, *Музичко-педагошке дилеме и теме*, Нови Сад: Академија уметности у Новом Саду.

Rajić (2019): С. Рајић, *Математика и музика у функцији дечјег развоја*, докторска дисертација. Retrieved in September 2022 from https://nardus.mpn.gov.rs/bitstream/id/16730/Disertacija.pdf

Semoz (2020): М. Семоз, Педагошке импликације различитих приступа процењивању квалитета знања ученика, *Зборник радова*, год. 23, бр. 22, Ужице: Универзитет у Крагујевцу, Педагошки факултет у Ужицу, 131–152.

Swe Khine, Areepattamannil (Eds.) (2019): M. Swe Khine, S. Areepattamannil, *STEAM Education: Theory and Practice*, Switzerland: Springer.

Tabašević et al. (2021): M. Tabašević, D. Lakušić, N. Kuzmanović, S. Vukojičić, M. Glišić, S. Jovanović, Ruderal vegetation in Serbia – diversity and floristic composition, *Botanica Serbica*, 45(2), Belgrade: University of Belgrade, Institute of Botany and „Jevremovac" Botanical Garden, 251–261.

UNESCO (2012): *Arts Education, from diversity to sustainability*, First Celebration of International Arts Education Week.

Vasiljević (2006): З. Васиљевић, *Методика музичке писмености*, Београд: Завод за уџбенике и наствна средства.

Vasiljević (1999): З. Васиљевић, *Теорија ритма са гледишта музичке писмености*, Београд: Универзитет уметности у Београду.

Vujošević (2017): N. Vujošević, Uticaj vizuelnih signala na segregaciju melodije kod muzičara i nemuzičara, *Зборник радова са XI међународног научног скупа*, књига III, Крагујевац: Филолошко-уметнички факултет, 83–91.

Yakman (2010): G. Yakman, *What is the Point of STEAM? A Brief Overview of STEAM Education*, Retrieved in September 2019 from https://www.academia.edu/8113832/What_is_the_Point_of_STEAM_A_Brief_Overview_of_STEAM_Education

Ивана М. Милић

Универзитет у Крагујевцу
Факултет педагошких наука у Јагодини
Катедра за дидактичко-методичке науке

Јелена М. Младеновић

Катедра за природно-математичке и информатичке науке

ОД МОРФОЛОГИЈЕ БИЉАКА ДО РИТМИЧКИХ ОБРАЗАЦА (У МУЗИЦИ): STEAM ПРИСТУП ПРОУЧАВАЊУ РЕЛАЦИЈА У МАТЕМАТИЦИ

Резиме: У овом раду су са теоријског аспекта сагледани иновативни искораци у раду са студентима, применом STEAM приступа, и могућности унапређивања квалитета универзитетског образовања. Модернизација универзитетске наставе подразумева праћење глобалних трендова, при чему је примарни циљ формирање свестраног, компетентног студента, који ће моћи да одговори савременим захтевима друштва, и да учествује у размени информација у вези са актуелним научним достигнућима уз стално јачање сопствених капацитета. Током универзитетског образовања, потребно је да студенти развију своје потенцијале и компетенције, формирају позитивне ставове према будућој васпитачкој професији, и схвате значај улоге васпитача приликом одабира приступа васпитно-образовном раду.

Компетенције студената, будућих васпитача, испољавају се кроз реализацију активности и огледају се у активирању дечјих потенцијала, обогаћивању дечјих искустава, подстицању креативности и приближавању научних чињеница (појава и процеса) и феномена у области природних наука и музичке уметности. Управо је зато важно студенте упознати са иновативним приступима васпитно-образовном раду, јер ће стечена знања моћи да трансформишу и правилно примене касније у свом будућем раду како би обезбедили холистички развој деце.

У раду је представљен пример активности где су интегрисани садржаји три различите области – математике, природних наука и музичке уметности, а све у циљу унапређења универзитетске наставе кроз различите приступе. Примери приказани у раду могу се применити током образовања студената, будућих васпитача, чиме би се ојачале њихове компетенције за рад са децом предшколског узраста.

Кључне речи: ритам, ботаника, математичке релације, STEAM приступ, студенти-васпитачи у предшколским установама.

Marija V. Milinković
Primary School “Desanka Maksimović“
Belgrade

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EVALUATION OF MOBILE APPLICATIONS IN THE TEACHING OF GEOMETRY

Abstract: Today’s world implies more and more frequent use of smartphones and their applications in every place and at every moment. In this paper, we will first talk about mobile educational applications in general, and then we will present various research related to the evaluation of mobile educational applications in the teaching of geometry (*Bos*, *Dick*, *Larkin*). Analyzing the relevant literature, the conclusion is drawn that the evaluation of mobile educational applications is important for learning if geometrical content can be successfully carried out according to three aspects: pedagogical, mathematical and cognitive. The pedagogical aspect implies the effectiveness of the application to assist in learning; the mathematical aspect of mobile educational applications is very often not fully satisfied because the incorrect use of mathematical language is noticeable, as well as the incorrect classification of shapes and objects; the cognitive aspect determines to what extent the application affects the development of students’ thought processes. According to previous research, mobile educational applications such as *Co-ordinate Geometry*, *Transformations* and *Attribute Blocks* were rated highly in all mentioned aspects of the evaluation.

Keywords: mobile educational applications, evaluation, mathematics teaching, geometric content.

INTRODUCTION

The times we live in present numerous challenges in terms of the use of digital technologies by both adults and younger populations. Digital technologies have become a key link in education but also in other areas of work. Today, young people spend most of their free time using computers, tablets, mobile phones and television, which increasingly affects the transition from the traditional dimension of education to a system of modernized education based precisely on the use of digital technologies. We will not leave out the fact that ”the role of digital technologies in the understanding of studied phenomena is not to replace natural and/or social reality and active learning of teaching content. Technology is an addition that gives a new dimension to learning and teaching“ (OECD, according to: Ristić,

marija.milinkovic90@gmail.com

Blagdanić 2017: 4). Namely, digital competences are one of the eight key ones in lifelong education prescribed by the European Union in order to respond to the constant progress and development of society (The European Parliament and the Council of the European Union 2006). ”Digital competence refers to the ability to safely and critically use information and communication technology (ICT) for work, in personal and social life, as well as in communication. Its key elements are basic ICT skills and abilities: the use of computers for finding, evaluating, storing, creating, displaying and exchanging information, as well as developing collaborative networks via the Internet“ (Ristić, Blagdanić 2017: 3). The functioning of today's society is unthinkable without the use of mobile phones, especially when it comes to young people. Mobile technologies represent portable devices that consist of hardware (physical parts) and software (operating systems and mobile applications) and enable communication through network services (Jarvenpaa, Lang 2005). Mobile technologies are also highly favored in the teaching process, because they provide a wide range of necessary information that makes the entire teaching process more qualitative and functional (Larkin 2014; Clement 2019; Juandi et al 2021). In this paper our focus will be specifically directed to the evaluation of mobile educational applications (APP) in teaching mathematics with special reference to their application when studying geometric content.

BRIEFLY ABOUT MOBILE APPLICATIONS

In order to explain the concept of ”mobile educational applications“, we will first explain what ”mobile learning“ means. Mobile learning is about sharing information through mobile technologies. It is a subtype of electronic learning where communication takes place through mobile phones instead of using a computer (Nordin, Embi, Yunus 2010). With the help of mobile learning, it is very easy to get necessary information through, for example, online dictionaries, various social media sites, voice search, etc. Mobile devices make this possible with the help of their touch screens, easy access to Internet browsers, and the use of microphones and cameras (Haag, Berking 2019). By using mobile phones, along with all the possibilities they provide, it is very easy to access certain mobile applications that aim to improve and create quality, functional knowledge. Research conducted in Australia and China confirmed the positive effects of using mobile phones in class and showed that students are far more motivated and interested in participating in the learning process (Zhang 2019). Mobile applications are software designed to provide a variety of uses on both mobile phones and tablets (Clement 2019). Mobile educational applications have been recognized as some of the most important innovations that have influenced teaching and learning, so there is an increased research interest on the introduction and implementation of mobile learning in the context of formal education (Panteli, Panaoura 2020). The rapid development of science and technology has been accompanied by the development of mobile educational applications that are mostly free and very easy to use and install. The use of mobile educational applications allows students to engage in problem-solving based learning activities, to work on tasks that are goal-oriented and to develop their own understanding through active involvement and sense-making (Charles-Owaba, Ahiakwo 2021). That mobile educational applications improve and enrich student's knowledge is also confirmed by the analysis of the mobile application called *Financial Maths* App, which was designed so that the student independently accesses the mathematical content, where the application explains each step in detail and motivates the student to think critically and creatively while solving problems that have real contexts. This application offers the possibility to engage with different concepts that lead to the solution of the problem. The application proved to be very acceptable to both teachers and students, offering the possibility of further development (Jordaan, Laubscher, Blignaut 2017).

EVALUATION OF MOBILE EDUCATIONAL APPLICATIONS IN ELEMENTARY GEOMETRY TEACHING

As the use of mathematical applications in classrooms becomes more frequent, research into their effectiveness is necessary to discover the best way to use them. This is especially true for geometry applications where accurate and dynamic representations are crucial in enhancing mathematics learning. Early findings indicate that most apps are limited in their ability to help students develop an understanding of geometric concepts. In this section of the paper, we will deal with the evaluation of educational software in order to examine qualitative evaluations of geometric applications based on pedagogical, mathematical and cognitive aspects.

Early research findings indicate that most graded geometry apps do little to help students develop understanding of geometric concepts and that accuracy in representations is not evident. Although research has been conducted on the mathematical effectiveness of applications (Attard, Curry 2012; Larkin 2013; Moyer-Packenham et al. 2015; Panteli, Panaoura 2020), there has not been much research on their usefulness in developing geometric concepts, but rather their basic descriptions. An initial review of applications (Larkin 2013) found few applications that are specifically geometric. However, the application market has progressed, that is, a lot of geometric applications have been made. According to data from 2015, there were about 150,000 educational applications in the *iTunes* store (148AppsBiz 2015). According to the latest data, there are over 520,000 educational applications (Pocketgamer.biz 2022).

Larkin's review of 53 geometry applications, published in the journal *Australian Primary Mathematics Classroom*, confirmed the findings of previous research on number and algebra applications. Namely, finding an adequate geometric application that is useful in elementary mathematics teaching is a difficult task, in terms of the time it takes and the poor quality of the applications that are available for download (Larkin 2014). During the search for mathematical applications, the following terms were used: geometry elementary education; geometry junior education; geometry primary education.

Dick criticized applications from a mathematical, pedagogical and cognitive perspective (Dick 2008). Dick suggests that students are most likely to describe the pedagogical value in terms of how it enabled them to interact with mathematics (for example, ”I made this triangle“, not just as a description of procedures to use, e.g., ”I adjusted the settings“). Therefore, in order for an application to be an effective tool, it must support any student action that will lead to a conceptual understanding of the underlying mathematical principle.

Dick suggests that pedagogical aspects relate to the effectiveness of digital tools to support learning and include ”the extent to which teachers and students believe that digital teaching tools enable students to engage with mathematics in ways that are appropriate to the nature of mathematical learning“ (Zbiek, Heid, Blume, Dick 2007). The effectiveness of digital teaching tools in terms of the pedagogical aspect must support the way in which students initially develop conceptual knowledge and later procedural and declarative knowledge. For example, the *Co-ordinate Geometry* app develops application-based learning by having students learn new concepts, apply these concepts, and then test their knowledge of what they have learned through a quiz (Larkin 2016).

Another aspect that is considered is the mathematical aspect. The mathematical aspect is present when the student's activity is ”probable, concrete and related to how mathematics is a functional part of life“ (Bos 2009: 171). It is defined as ”the devotion of digital teaching aids in showing mathematical properties, conventions and behavior as would be understood or expected by the mathematical community“ (Zbiek, Heid, Blume, Dick 2007: 1173). Dick warns that the desire to adapt the application to students and teachers can sometimes be contrary to correct mathematical structures (Dick 2008). Problems of the mathematical aspect (Larkin 2013) are generally related to the incorrect use of mathematical language or the classification of shapes and objects (e.g. checkers instead of rhombuses, squares are not considered quadrilaterals, triangles are not classified as polygons, and the lack of connection between mathematics and the real environment, with minor exception of the applications *Geometry 4 Kids* and *Simitri*).

The notion of cognition is crucial in geometry applications. The digital nature of the ”app object“ (Larkin 2013) potentially leads to a high level of cognitive development; for example, 3D objects can be disassembled and reassembled, and this can strengthen the connection between 3D objects and their 2D representations (e.g. mesh cube). The cognitive aspect implies acting on the rational side of the child's personality, strengthening knowledge, the need for learning, teaching and understanding the process of education (Suzić 2001). According to Bos, it is the degree to which the application helps the development of thought processes in students (Bos 2009).

According to Zbiek et al., the cognitive aspect refers to ”the ability of digital tools to reflect student's thought processes” (Zbiek, Heid, Blume, Dick 2007). In her research (Bos 2009), Bos categorized software according to the low, medium and high level of presence of these three aspects. In each dimension, it uses numerical values to represent the degree to which these three aspects are present. In order to make comparisons between the three aspects, numerical values are given from 1 (low level) to 10 (very high level) for each of the three aspects.

Table 1. Aspects in applications by level according to Bos (Bos 2009)

Aspects	Low level (1–3)	Medium level (4–7)	High level (8–10)
Pedagogical Aspect The extent to which the application can be used to support learning	It is hard to work on the app. Access to the application is difficult. The application is not suitable for mathematical content.	Using the app is not intuitive at first, but it becomes with practice. The presented mathematical contents are suitable but can be developed without the application.	Handling the application is intuitive and encourages user participation. Little or no training or instruction is required.
Mathematical Aspect The extent to which an application reflects mathematical properties, conventions, and behaviors	Mathematical contents are not sufficiently developed or are too complex. Not enough templates. There is no connection between mathematics and the real environment.	The application of mathematical content is unclear. The creation of a pattern is obvious, but it cannot be predicted or is unclear. There is a certain connection between mathematics and the real world.	The developed mathematical content is accurate and age appropriate. Patterns are accurate and predictable. Clear connection of mathematics with the real environment.
Cognitive aspect The degree to which the application helps develop the student's thought processes	There are no opportunities to explore or test assumptions. Static or inaccurate displays. Templates are not related to concept development.	Limited opportunities to explore or test assumptions. Minor glitches with the renderings, but still make sense. Limited connection between templates and concept development.	The app encourages exploration and testing assumptions. The displays are accurate and easy to navigate. Templates clearly help concept development.

In his research evaluating 53 geometric applications, Larkin used Bos's (Bos 2009) framework for evaluating educational software. The geometric application *Transformations* is an example of the fact that the design of the application requires additional help from an adult when using it, especially in the quiz part, but also to encourage the learning of mathematical content. The app is good in the research part but too complex in the quiz part. The app develops concepts very clearly – much more effectively than paper and pencil would. Mathematical content is correct, age-appropriate and accurate. There are no connections between mathematical examples and the real world. Research is encouraged and contributes to conceptual development.

The next application whose review we considered in this part is related to geometric shapes (*3D GeometryBasica*). The application includes eight 3D objects. The only action that can be performed is zooming in or out to make the object larger or smaller. Each subject has a mathematical description and symbols and formulas for calculating area and volume. Reviewer comments say that using the app is intuitive, mostly due to its limited options, but that the content is accurate. From a conceptual development perspective, the application contains complex formulas for calculating surface area and volume, but no relationship is established between the surface area and volume of objects or between the surface areas and volumes of different objects. The application does not have examples of the connection of mathematics with the real environment. The app has very limited utility and does not do anything that other manipulatives or even pen and paper cannot already do.

Next, the *Shape Rotate* app was rated low because instead of students specifying how to draw specific angles, the app allows them to enter a numerical value, and then the app draws the angle for them. Given that many applications are made by non-educators, poor mathematical structuring of future applications is likely to continue (Larkin 2016).

The most popular area is geometric shapes and this may be because these applications are easy to make from a technical perspective. Although they are the most common, most of these shape apps are very basic and only involve naming shapes and very simple matching exercises. Many of these activities can be done more easily using the right objects. Apps related to angles and 1D geometry were frequent, but this is due to the large number of quiz apps, not the availability of a large number of apps that develop an understanding of 1D and angles.

Less than half of all evaluated apps (26 out of 53) failed to get a six in any of the three aspects (does not support pedagogy, is not mathematically correct and is cognitively inactive). The mean score of 53 applications (12.9/30) did not reach a passing grade. In short, mathematical, pedagogical and cognitive aspects are poorly represented (Larkin 2016). Also, applications that received a score of 6 or higher scored well in terms of the pedagogical aspect but not so well in terms of the mathematical and cognitive aspects. However, many applications met only one pedagogical criterion: they are easy to use without instructions. Given that applications are made by people who are not mathematicians, it is not surprising that this aspect is the most prevalent in applications. Other applications partially meet the criteria of developing ideas and concepts about basic geometric figures in an appropriate way, without having to do anything more than what could easily be displayed on an interactive whiteboard or using some other teaching aids. Although some of the apps scored highly in one of the aspects, they did not score highly in other aspects because they had a weak connection between geometry and the real environment as experienced by children and were ultimately inconsistent in terms of higher levels of abstraction (e.g. squares are not classified as quadrilaterals or triangles are not included in polygons). In mathematics, concepts are much more abstract than those in everyday life, and learning itself has the characteristics of more abstraction. More abstract means more distant from perceptions and concrete impressions. For example, square < rectangle < quadrilateral < polygon (Đokić 2007). Table 2 summarizes the seven applications that were rated six or higher in all three aspects.

Table 2. Applications that were rated with a score of six or higher in all three aspects (Larkin 2016)

Name of the application	Pedagogical Aspect /10	Mathematical Aspect /10	Cognitive aspect /10	Total score /30
Co-ordinate Geometry	9	8	9	26
Transformations	9	8	9	26
Attribute Blocks	8	8	8	24
Shapes–3D Geometry	9	6	8	23
Shapes and Colours	7	6	7	20
Pattern Shapes	8	6	6	20
Isometry Manipulative	7	6	6	19

It should be noted that only one application, *Simitri* (Simitri 4, 9, 8), received a very low rating from the pedagogical aspect and high ratings from the mathematical and cognitive aspects. Therefore, students should not use the app alone, without supervision. Except in the case of the top three apps, teachers must determine the exact purpose of using the app and then look at the content covered as well as the ratings of all aspects to find an appropriate one that supports students' mathematical learning.

The application *Geometry Montessori* (Geometry Montessori 9, 6, 5) is rated the same or better than the three applications that are among the top seven, but it is relatively poor when looking at the cognitive aspect. The application *Geometry Montessori* would be appropriate to use for the review of the material because it received a rating of 9 from the pedagogical aspect but not for developing the mathematical or cognitive aspect.

For example, the *Pattern Shapes* app made Larkin's list because it scored at least a six in each of the categories. The app is really useful in a pedagogical sense (score 8), but it does not support connecting examples from everyday life. This pattern of quality in one area and weakness in one or both of the remaining two is also present in other applications, which means that teachers need to do significant prior planning if they want the application to be useful and not potentially harmful to some forms of mathematical knowledge.

Ristić and Blagdanić (2017) present a broader proposal when it comes to the evaluation of mobile applications, and it is about analyzing applications from a point of view that includes six criteria: (1) scientific and professional criteria; (2) pedagogical-psychological and didactic-methodical criteria; (3) ethical criteria; (4) language criterion; (5) technological and graphic criterion and (6) security criterion.

In the following text, we will briefly explain what each of these criteria entails. First, the scientific-professional criterion implies that the application must enable the achievement of the goals and objectives of the subject, the contents must be provided by the curriculum, and the application must be harmonized with the methodology. Pedagogical-psychological and didactic-methodical criteria imply that the application must be suitable for the age of the students, encourage students to be active and engage in cooperative learning, develop independence and initiative in learning, encourage different forms of learning, ensure interactivity and feedback, etc. Ethical criteria include encouraging tolerance and respect for diversity, promoting non-violence, and respect for inclusion and gender equality. The language criterion includes respect for the language norms of the native language or a national minority or a foreign language; the language and sentences must be adapted to the age of the students, as well as the professional terminology used. The technical and graphic criterion refers to compliance with technological W3C standards; the application must have clear and simple navigation and instructions that facilitate use for both students and parents and teachers, and graphic and multimedia elements must be of high quality, clear, content-related and accompanied by a title or explanation. The security criterion implies the safe transfer of data from and to users, and students must not be led to activities that could put them in danger.

It is widely accepted in the mathematics community that if used thoughtfully, digital tools can enhance mathematics learning (Burns, Hamm 2011; Carbonneau, Marley, Selig 2013; Moyer-Packenham et al. 2015; Larkin 2016; Charles-Owaba, Ahiakwo 2021; Yosiana, Djuandi, Hasanah 2021), but teachers still play a key role in deciding how and when to use apps. More significant reviews of geometry applications will be needed in the future, and Bos's (Bos 2009) software categorization and three aspects of application quality considerations (Dick 2008) may be useful in order to support students' mathematical learning.

CONCLUSION

Since we live in a time where children are using smartphones at an ever earlier age, it is clear that the application of mobile learning is a sign of the future and will be an increasing support for education. The young population is increasingly using smartphones for the purpose of obtaining various information through social networks, mobile applications, etc., and what is particularly important to them is the availability and use of these devices anytime and anywhere. Given that the educational system is increasingly based on digital technologies, the use of mobile phones and applications is therefore completely justified. By using mobile applications, students can get very high-quality and effective knowledge, and in order to achieve that, all those applications must be evaluated from several aspects so that the result of their use is truly effective. In this paper, we have considered the evaluation of mobile educational applications according to different aspects, while highlighting the pedagogical, mathematical and cognitive ones. If the application meets the requirements of the pedagogical aspect, students should first develop conceptual, then procedural and finally declarative knowledge. Within the mathematical aspect, the mathematical content in the application must be accurate and adapted to the age of the students, and the abstract world of mathematics must be related to the real environment. The cognitive aspect involves encouraging and developing thought operations through the use of the application, so all templates in the application must be clear and correct. Therefore, the evaluation of the mobile educational application from the aforementioned aspects can greatly contribute to the creation of quality and lasting knowledge among students. Through this work, we got acquainted with the various advantages and weaknesses of mobile educational applications and the learning of geometric content. Therefore, before using any application that is used in the teaching process, it should always be evaluated. Specifically, when it comes to learning geometric content, the mobile educational applications that are rated very highly from a pedagogical, mathematical and cognitive point of view are *Co-ordinate Geometry*, *Transformations* and *Attribute Blocks*. In some future research, it would be challenging to evaluate these applications according to the criteria (6 criteria) proposed by Ristić and Blagdanić.

REFERENCES

Attard, Curry (2012): C. Attard, C. Curry, Exploring the use of iPads to engage young students with mathematics, In: J. Dindyl, L. P. Cheng, S. F. Ng (Eds.), *Mathematics education: Expanding horizons Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia*, Singapore: MERGA, 75–82.

Bos (2009): B. Bos, Virtual math objects with pedagogical, mathematical and cognitive fidelity, *Computers in Human Behavior*, 25(2), 521–528.

Burns, Hamm (2011): B. A. Burns, E. M. Hamm, A comparison of concrete and virtual manipulative use in third and fourth grade mathematics, *School Science and Mathematics*, 111(6), 256–261.

Carbonneau, Marley, Selig (2013): K. J. Carbonneau, S. C. Marley, J. Selig, A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives, *Journal of Educational Psychology*, 105(2), 380–400.

Charles-Owaba, Ahiakwo (2021): T. Charles-Owaba, M. J. Ahiakwo, Effects of Teaching Geometry Using Mobile App Instructional Techniques on Secondary School Students' Academic Achievement in Bayelsa State, *Journal of Mathematical Sciences & Computational Mathematics*, 2(3), 404–421.

Clement (2019): J. Clement, Mobile app usage – Statistics & Facts [online], *Statista*. Retrieved in Jun 2021 from https://www.statista.com/topics/1002/mobile-app-usage.

Dick (2008): T. P. Dick, Fidelity in technological tools for mathematics education, In: G. W. Blume, M. Kathleen Reid (Eds.), *Research on technology and the teaching and learning of mathematics*, Vol. 2, 333–339.

Haag, Berking (2019): J. Haag, P. Berking, Design Consideration for Mobile Learning, In: Y. A. Zhang, D. Cristol (Eds.), *Handbook of Mobile Teaching and Learning*, Singapore: Springer, Chapter 13.

Jarvenpaa, Lang (2005): S. L. Jarvenpaa, K. R. Lang, Managing the paradoxes of mobile technology, *Information systems management*, 22(4), 7–23.

Juandi et al. (2021): D. Juandi et al., The Effectiveness of Dynamic Geometry Software Applications in Learning Mathematics: A Meta-Analysis Study, *International Journal of Interactive Mobile Technologies (iJIM)*, 15(02), 18–37.

Jordaan, Laubscher, Seugnet Blignaut (2017): D. Jordaan, D. Laubscher, A. Seugnet Blignaut, Design of a prototype mobile application to make mathematics education more realistic, In: I. A. Sanchez, P. Isaias (Eds.), *13th International Conference Mobile Learning 2017*, International Assn for Development of the Information Society, 3–11.

Larkin (2013): K. Larkin, Mathematics Education. Is there an App for that?, In: V. Steinle, L. Ball, C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia)*, Melbourne: MERGA, 426–433.

Larkin (2014): K. Larkin, Ipad Apps that promote mathematical knowledge? Yes, they exist!, *Australian Primary Mathematics Classroom*, 19 (2), 28–32.

Larkin (2016): K. Larkin, Finding quality geometry apps, *Australian Primary Mathematics Classroom*, 21(4), 22–26.

Moyer-Packenham et al. (2015): P. S. Moyer-Packenham et al., Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps, *Journal of Computers in Mathematics and Science Teaching*, 34(1), 41–69.

Nordin, Embi, Yunus (2010): N. Nordin, M. A. Embi, M. Yunus, Mobile Learning Framework for Lifelong Learning, *Procedia Social and Behavioral Sciences*, 7(C), 130–138.

Panteli, Panaoura (2020): P. Panteli, A. Panaoura, The effectiveness of using mobile learning methods in geometry for students with different initial mathematical performance, *Social Education Research*, 1–10.

Ristić, Blagdanić (2017): M. Ristić, S. Blagdanić, Nove perspektive u obrazovanju – vanučionička nastava u digitalnom okruženju, *Inovacije u nastavi*, XXX(2), Beograd: Učiteljski fakultet, 1–14.

PocketGamer.biz (2022): App store metrics. Retrieved in October 2022 from App Store Metrics | Pocket Gamer.biz | PGbiz.

Suzić (2001): N. Suzić, *Interaktivno učenje III*, Banja Luka: Teacher's Training Center.

The European Parliament and the Council of the European Union (2006): *Recommendation of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning*. Retrieved in January 2022 from http://data.europa.eu/eli/reco/2006/962/oj.

Yosiana, Djuandi, Hasanah (2021): Y. Yosiana, D. Djuandi, A. Hasanah, Mobile learning and its effectiveness in mathematics, *Journal of Physics: Conference Series*, International Conference on Mathematics and Science Education (ICMScE), Indonesia, 1806, 1–29.

Zbiek, Heid, Blume, Dick (2007): R. M. Zbiek, M. K. Heid, G. W. Blume, T. P. Dick, Research on technology in mathematics education: A perspective of constructs, In: F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, 1169–1207.

Zhang (2019): Y. A. Zhang, Characteristics of Mobile Teaching and Learning, In: Y. A. Zhang, D. Cristol (Eds.), *Handbook of Mobile Teaching and Learning*, Singapore: Springer.

Đokić (2007): O. Đokić, *Pojam linije u početnoj nastavi geometrije*, Beograd: Učiteljski fakultet.

148AppsBiz. (2015): App store metrics. Retrieved in January 2019 from http://148apps.biz/app-store-metrics/.

Марија В. Милинковић

Основна школа „Десанка Максимовић"
Београд

ЕВАЛУАЦИЈА МОБИЛНИХ АПЛИКАЦИЈА У НАСТАВИ ГЕОМЕТРИЈЕ

Резиме: Данашње време подразумева све више и све чешће коришћење паметних телефона и њихових апликација на сваком месту и у сваком тренутку. У овом раду прво ћемо говорити уопштено о мобилним образовним апликацијама, а потом ћемо представити различита истраживања која су у вези са евалуацијом мобилних образовних апликација у настави геометрије (*Bos*, *Dick*, *Larkin*). Анализирајући релевантну литературу изводи се закључак да се вредновање мобилних образовних апликација које су значајне за учење геометријских садржаја успешно може извршити са три аспекта: педагошког, математичког и когнитивног. Педагошки аспект подразумева ефикасност апликације да помогне у учењу, мобилне образовне апликације врло често не задовоље у потпуности математички аспект, јер је приметно погрешно коришћење математичког језика, али и погрешна класификација фигура и тела, док се кроз когнитивни аспект утврђује у којој мери апликација утиче на развој мисаоних процеса ученика. Према досадашњим истраживањима мобилне образовне апликације као што су *Co-ordinate Geometry*, *Transformations* и *Attribute Blocks* оцењене су високим оценама у свим споменутим аспектима евалуације.

Кључне речи: мобилне образовне апликације, евалуација, настава математике, геометријски садржаји.

REVIEWS

Zuzana Hucki
School of Science & Technology
Nottingham Trent University
United Kingdom

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BOOK REVIEW: *NEW YEAR'S PRESENT FROM A MATHEMATICIAN* BY SNEZANA LAWRENCE

When I first received the book during the lockdown, I was slightly disappointed, as I was expecting a large book. My disappointment faded very quickly when I realised that the book is jam-packed with information, stories, and notes for further research.

The book is not chronological in an ordinary sense. The author has lifted us into the fourth dimension and connected different mathematicians through historical facts, birthdates, and death anniversaries organised in chapters covering every month from January to December.

Mathematics is metaphorically represented as a desert, and mathematical stories are represented as drops of wisdom. While the author is motivated by a personal experience of being lost in the desert, I could not help thinking of the quote from *The Little Prince*: "'What makes the **desert** beautiful,' said the **little prince**, 'is that somewhere it hides a well…' "Mathematical discoveries presented in this book are like wells discovered in a desert.

January's chapter starts with Newton, one of the greatest British scientists and mathematicians, and finishes with the poem "The Newtonian system of the world, the best model of government: an allegorical poem", which made me wonder: did any other mathematician have a poem written about them?

February's chapter borrows the date of completion (February, 532 AD) of the Hagia Sophia, a Christian cathedral in Istanbul, is filled with geometry and the story of the Greek mathematician Thales. As we would not know Thales's date of birth or death, the author chose a topic very close to her heart, architecture, and filled the chapter with the connections between maths and architecture.

A similar connection is the main topic of the chapter "March": Christopher Wren, the mathematician and architect put in charge of St. Paul's restoration after the Great Fire of London. The chapter on March, like the beginning of spring, is filled with beauty. Beautiful drawings of different curves and the answer to the question, "Beauty is in the eye of beholder – or in Mathematics?" It also includes the story of how mathematicians proposed the questions and solved the problems

zuzana.hucki@ntu.ac.uk

of catenary curves. The author states that "before the internet, search engines, and social media, scientific news still managed to travel… mathematicians from around Europe corresponded and exchanged ideas and often posed to each other their mathematical questions, problems, and challenges".

The following two months are my favourite. They both celebrate female mathematicians.

"April" celebrates the amazing mathematical brain of Emmy Noether who is creditedd as the mother of modern algebra. Lawrence compares modern algebra to modern art. Did you know that ideals in abstract algebra are special types of rings, subsets that are closed with respect to the "multiplication" operation of the ring?

The chapter on May brings the wonderful story of the first female mathematics professor, Italian Maria Agnesi and the discovery of another type of curve named the 'Witch of Agnesi.' This chapter also uncovers the story of the book *Newtonianism for ladies*, written by Francesco Algarotti, which was a popular science bestseller in the 18th century; it was translated into English too. The author is suggesting that we should forgive the patronising and dismissive tone of the writing, considering that science and maths needed to be explained differently to females. It brought much good by spreading Newtonian science throughout society, allowing Agnesi to become a mathematician and her discoveries to be recorded as a part of the history of mathematics.

Do not miss the chapter on September and the story of Paul Erdos, a true citizen of the world. Discover why he would greet his co-researcher hosts with "my brain is open" and learn about his amazing generosity through offering rewards for unsolved mathematical problems.

I really enjoyed the author's rich, poetic style of writing.

For future books, I would urge the author to expande her metaphorical desert of discovery to include the history of mathematics from the rest of the world, less known and explored than Europe's.

To find the explanation for the unusual title of the book you need to read December's chapter and discover what a mathematician's present to a friend, who loves maths, could be.

This book is a wonderful present which keeps giving even after multiple readings.

Milan P. Milikić
University of Kragujevac
Faculty of Education in Jagodina
Department of Didactics and Methodology

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A CONTRIBUTION TO THE METHODOLOGY OF TEACHING MATHEMATICS

(Mirko Dejić, Milana Egerić and Aleksandra Mihajlović, *Methodology of teaching mathematics in lower grades of elementary school*, Jagodina: Faculty of Education, University of Kragujevac, 2022, 500 pages)

Methodology of teaching mathematics in lower grades of elementary school (2022) is the second edition of the textbook authored by prof. Mirko Dejić, Ph.D., retired full professor of the Teacher Education Faculty of the University of Belgrade, prof. Milana Egerić, Ph.D., retired full professor of the Faculty of Education of the University of Kragujevac and prof. Aleksandra Mihajlović, Ph.D., associate professor of the Faculty of Education of the University of Kragujevac. The first edition of the textbook was published in 2015, and the publisher is the Faculty of Education of the University of Kragujevac headquartered in Jagodina. The second edition of the textbook contains 500 pages of text within which, among other things, there are 202 pictures, with the main goal of providing readers with a better insight into the theoretical considerations and facilitating the monitoring of the presented content. In addition to the Preface, in which the authors provide basic information and emphasize its primary purpose as an educational resource for pre-service class teachers, the textbook contains eighteen chapters numbered in Roman numerals. At the beginning of each chapter, there are quotes from famous mathematicians to encourage readers to think about mathematics in a new way. At the end of the textbook, in the Literature section, the authors provide a list of 120 bibliographic items.

The first chapter, "Mathematics as a scientific discipline and teaching subject", is structured so that it consists of several sections. The aim of this chapter is to familiarize readers with the general characteristics of mathematics as a science on the one hand, and as a teaching subject on the other hand, through a review of the historical development of mathematics. Within the chapter, the authors outline the periodization of the historical development of mathematics and define the terminology used in various areas of initial mathematics education. Special emphasis

milikic.milan@yahoo.com

is placed on the subject and definition of mathematics and to what extent they are influenced by the discipline's historical development.

The content of the second chapter, "Methodology of teaching mathematics as a scientific and study discipline", is also divided into sections within which the authors provide conceptual definitions of methodology of teaching mathematics and methodology of teaching mathematics in lower grades of elementary school, differentiating the former from the latter. They define the problem and subject of the study of methodology of teaching mathematics in lower grades of elementary school and indicate its relationship with other sciences. They emphasize the connection of methodology of teaching mathematics with mathematics, pedagogy, psychology, logic, and philosophy, emphasizing the multidisciplinary character of methodology of teaching mathematics. In the last section of this chapter, the subject and goal of methodology of teaching mathematics as a discipline of study are defined, primarily focusing on the training of students for independent preparation and practical implementation of teaching mathematics in the lower grades of elementary school.

In the chapter "Psychological and logical foundations of teaching mathematics", the authors start with the definition of mathematical thinking and then present two theories (Piaget's and Vygotsky's) of the development of children's thinking, describing the main characteristics of children's thinking at each stage of cognitive development. The authors place special emphasis on the mathematical concept andthe process of its forming and definition. In addition, with numerous examples, they describe the types of mathematical reasoning and the ways of performing mathematical proofs in lower grades of elementary school.

The fourth chapter, "Analysis and explanation of mathematical concepts formed in lower grades of elementary school", refers to all contents included in the initial mathematics education. In this chapter, the authors provide a general overview of the content on sets, arithmetic, algebra, geometry, as well as content related to measurement and measures. The chapter here presents the order of study, the scope in which the contents are adopted, and reveals the specifics of the contents for which methodical transformation pre-service class teachers should be trained.

From the fifth to the eleventh chapter, the authors provide a detailed methodical transformation of the mathematical contents included in the mathematics teaching in the lower grades of elementary school. Starting from the introduction of the concept of a set, through the formation of the concept of a number, basic calculation operations, spatial relations, fractions, equations and inequalities, geometric concepts, to the procedure of measurement and familiarization with the units of measurement, the authors strive to make learning processes clearer to pre-service class teachers and all those participants involved in the educational process. Along with numerous concrete examples, which help the authors suggest ways to introduce the mentioned concepts, detailed instructions are given and all steps in the process are explained, making it easier for students to prepare for teaching mathematics.

The aim of the twelfth chapter, “Arithmetic tasks in initial mathematics education”, is to indicate the place and role of tasks in teaching mathematics. In this chapter, the authors provide a definition of mathematical problems and introduce the reader to the types and structure of arithmetic problems. The authors paid special attention to methodical guidance when solving arithmetic tasks. They state the stages involved in solving arithmetic problems and the methods that can be used to solve them.

In the thirteenth chapter, “Motivating and encouraging the learning of mathematics” a particular focus is on the use of different ways to stimulate students’ interest in mathematics. In an illustrative way, with a large number of examples of tasks and mathematical and didactic games, the authors have shown how to positively influence students’ motivation to discover the world of mathematics.

The content of the fourteenth chapter, “Teaching (didactic) principles in initial mathematics education”, refers to the adaptation of general didactic principles to elementary mathematics teaching. The authors give explanations of the necessity of respecting the teaching principles in mathematics classes, simultaneously presenting numerous situations in which some of the principles are violated.

Within the chapter “Teaching methods and teaching systems in initial mathematics education”, the authors list the types of teaching methods and describe the ways of their application in different phases of the mathematics lesson. In addition to methods, special attention is paid to familiarizing pre-service class teachers with different didactic systems and approaches. They emphasize problem-based and programmed learning/instruction, differentiated learning, project-based learning, integrative teaching approach and active learning/teaching. Each of the teaching systems is described in detail by the authors through concrete examples of application in elementary mathematics education.

The sixteenth chapter, “Organization and implementation of initial mathematics teaching”, shows as its primary goal the process of familiarizing pre-service class teachers with the process of planning, preparing, and implementing mathematics lessons. In this chapter, the authors state clearly and concisely the types of lessons, describe the forms of work in the mathematics lesson, as well as the kind of structure the lesson should reflect. Through this chapter, the authors also aim to equip students for self-reflection on the lesson held and methodical analysis of the mathematics lessons of other students. Important parts of this chapter include the sections on the approaches to organizing extracurricular work and the way of working in combined classes, in which the authors demonstrate the basic principle of working in such classes with an example of instruction.

In the seventeenth chapter, “Assessment and evaluation of the students’ work in mathematics teaching”, the types, functions, and criteria of assessment and evaluation are listed. In this chapter, the authors give specific proposals and suggestions for how and on what the students should be evaluated. Also, the authors present some examples of standardized tests and tests with tasks at different levels of educational achievement for certain thematic units in order to make it easier for pre-service class teachers to independently create such tests.

In the last, eighteenth chapter, “Examples of lesson plans”, there are models of mathematics lesson plans that should facilitate and prepare students for planning, preparing, and conducting lessons in practice schools.

In the textbook *Methodology of teaching mathematics in lower grades of elementary school,* the authors uniquely provide guidelines to understand the problem of systematic methodical education of pre-service class teachers. With numerous examples and detailed methodical guidance in a graphic and simplified way, they make the readers better understand the importance of adequate skills for teaching mathematics at the earliest school ages. At the end of each chapter, the authors provide a large number of questions and tasks for independent work, thus enabling pre-service class teachers to research further and look for new solutions. Additionally, at the end of each chapter, there are instructions on which tasks in the workbook (Practicum) should be done to better understand the content presented in the corresponding chapter.

On the basis of everything previously stated, we can conclude that the textbook *Methodology of teaching mathematics in lower grades of elementary school*, authored by prof. Mirko Dejić, Ph.D., prof. Milana Egerić, Ph.D., and prof. Aleksandra Mihajlović, Ph.D., represents an important contribution to the methodology of teaching mathematics.

УПУТСТВО АУТОРИМА

Узданица, часопис за језик, књижевност и педагошке науке, објављује научне и стручне чланке. У категорији научних чланака доноси оригиналне научне радове, прегледне радове, кратка или претходна саопштења, научне критике, односно полемике и осврте. У оквиру стручних чланака даје стручне радове, информативне прилоге и приказе.

Оригинални научни радови треба да садрже претходно необјављене методолошки утемељене резултате сопствених истраживања. Прегледни рад садржи оригиналан, детаљан и критички приказ истраживачког проблема. Кратко или претходно саопштење представља оригинални научни рад пуног формата, мањег обима или полемичког карактера. Научне расправе на одређену тему, засноване на научној аргументацији, дају се у оквиру научне критике, полемике и осврта.

У оквиру стручних прилога дају се стратегије и искуства корисна за унапређење професионалне праксе, уводници, коментари и прикази књига. Изузетно, у Часопису, примерено „Акту о уређивању научних часописа" Министарства за науку и технолошки развој Републике Србије, могу бити објављивани и монографски радови, као и критички прегледи научне грађе: историјско-архивске, лексикографске и библиографске.

Језик рада може бити српски и енглески, а према научној проблематици могу се објављивати и радови на другим језицима.

За објављивање у Часопису прихватају се искључиво радови који нису претходно објављивани. Све приспеле радове рецензирају два рецензента, после чега Редакција доноси одлуку о објављивању и о томе обавештава аутора у року од највише три месеца. Рукописи се шаљу електронском поштом, а прилози (цртежи, графикони, схеме) могу бити послати поштом и не враћају се. Адреса уредништва и електронска адреса дате су у импресуму часописа.

Рад приложен за објављивање треба да буде припремљен према стандардима часописа *Узданица* да би био укључен у процедуру рецензирања. Неодговарајуће припремљени рукописи неће бити разматрани.

Обим и фонт

Рад треба да буде написан у текст процесору Microsoft Word, фонтом Times New Roman величине 12 тачака, ћирилицом, са проредом од 1,5 реда. Обим оригиналних научних, прегледних и стручних радова је до једног ауторског табака (око 30.000 знакова), а информативних прилога и приказа до 1/3 ауторског табака (око 10.000 знакова). Уредништво задржава право да публикује и радове који прелазе овај обим.

Име аутора

Наводи се пуно име, средње слово и презиме, као и година рођења (свих) аутора. Година рођења се не објављује у *Узданици*, али се користи у бази аутора Народне библиотеке. Презимена и имена домаћих аутора увек се исписују у оригиналном облику (са српским дијакритичким знаковима), независно од језика рада.

***Назив установе аутора* (*афилијација*)**

Наводи се пун (званични) назив и седиште установе у којој је аутор запослен, а евентуално и назив установе у којој је аутор обавио истраживање. У сложеним организацијама наводи се укупна хијерархија (на пример: Универзитет у Београду, Филозофски факултет, Одељење за социологију).

Афилијација се исписује непосредно након имена аутора. Функција и звање аутора се не наводе.

Контакт подаци

Адреса или имејл-адреса аутора даје се у напомени при дну прве странице чланка. Ако је аутора више, даје се само адреса једног, обично првог аутора.

***Апстракт* (*сажетак*)**

Апстракт је кратак информативни приказ садржаја чланка који читаоцу омогућава да брзо и тачно оцени његову релевантност. Саставни делови сажетка јесу циљ истраживања, методе, резултати и закључак. Сажетак треба да има од 100 до 250 речи и треба да стоји између заглавља (наслов, имена аутора и др.) и кључних речи, након којих следи текст чланка.

Резиме

Ако је језик рада српски, сажетак на страном језику даје се у проширеном облику, као резиме. Посебно је пожељно да резиме буде у структурираном облику. Дужина резимеа може бити до 1/10 дужине чланка. Резиме се даје на крају чланка, након одељка Литература. Саставни део резимеа на страном језику чини и пуно име аутора, потпуна афилијација, назив рада и кључне речи.

Кључне речи

Број кључних речи не може бити већи од 10. У чланку се дају непосредно након сажетка, односно резимеа.

Литература

За навођење референци користи се АПА стил, а референце се могу наводити и по следећем моделу.

1. Књига

У тексту: (презиме година: страна)
У списку литературе: презиме (година): иницијал имена презиме, *наслов*, место: издавач.

Кристал (1999): Д. Кристал, *Енциклопедијски речник модерне лингвистике*, Београд: НОЛИТ.
Чомски (2008): N. Čomski, *Hegemonija ili opstanak*, Novi Sad: Rubikon.
Чомски (1968): N. Chomsky, *Language and Mind*, Harcourt, Brace and World: New York.

2. Чланак

У тексту: (презиме година: страна)
У списку литературе: презиме (година): иницијал имена презиме, наслов чланка, *наслов часописа/зборника*, број, место: издавач, страна.

Јовановић, Симић (2009): Ј. Јовановић, Р. Симић, Текст као лингвистичка и комуникацијска структура, *Српски језик*, XIV/1–2, Београд: Научно друштво за неговање и проучавање српског језика, 325–345.

Када се исти аутор наводи више пута, поштује се редослед година у којима су радови публиковани. Уколико се наводи већи број радова истог аутора публикованих у истој години, радови треба да буду означени словима уз годину издања, нпр.: 1999а, 1999б...

Навођење дела које има више од једног аутора подразумева да се имена аутора наводе према редоследу који је дат на насловној страни.

У тексту: (Франковић, Ракић, Вилотијевић 1973)
У списку литературе: Франковић, Ракић, Вилотијевић (1973): D. Franković, B. Rakić, M. Vilotijević, *Vaspitni rad u domovima*, Beograd: Delta pres.

Ако је више од три аутора, у тексту се наводи презиме првог аутора и додаје се „и др.", а у оквиру листе референци треба навести имена свих аутора према редоследу на насловној страни књиге/чланка.

Навођење необјављених радова није пожељно, а уколико је неопходно, треба навести што потпуније податке о извору.

Веб-документ

Презиме аутора, година, назив документа (курзивом), датум када је сајт посећен, интернет адреса сајта, нпр.:

Mercer, S. (2008): Learner Self-beliefs. *ELT Journal* 2008 62(2): 182–183. Retrieved in January 2009 from http://eltj.oxfordjournals.org/cgi/content/full/62/2/182

Цртежи*, *слике и табеле

Слике (цртежи, графикони, схеме) и табеле могу се припремити компјутерском или класичном технологијом (тушем или оловком на папиру). Дају се у посебном фајлу или на посебним папирима. У основном тексту се маркира место где долазе и не уводе се у текст. Табеле, слике и илустрације морају бити разумљиве. Нису пагиниране и морају имати редни број, наслове и легенде (објашњења ознака, шифара и скраћеница) класификоване по врстама и нумерисане унутар своје категорије. На папиру редни број слике или табеле, као и презиме аутора морају бити уписани на полеђини графитном оловком. Приказивање истих података табеларно и графички није дозвољено.

Статистички подаци дају се према параметрима научних методологија.

Уредништво

INSTRUCTIONS FOR AUTHORS

Uzdanica, an open access journal, is the journal for language, literature, art and education, publishes original research and review articles. In the category of the original research articles it publishes original research papers, review articles, short or previously presented reports, scientific reviews, and commentaries on topics of concern to the academic community. In the category of the review articles the journal distributes review articles, informative contributions and reviews.

Original research articles should be an original work based on the original and methodologically established research results, and should not have been accepted for publication elsewhere. Review articles should contain original, detailed and critically established research problem. Short or previously presented reports should present original, full-length scientific paper, in smaller volume or polemical character. Scientific reviews on certain topics, based on the scientific arguments, are published in the category of scientific review, polemics and brief notices.

Review articles should give strategies and experience useful for the professional practice improvement, editorials, and annotations or the reviews of books. Especially, according to the requirements of the “Act on the regulation of scientific journals” issued by the Ministry of Education, Science and Technological Development of the Republic of Serbia, the journal publishes monographic papers as well as critical reviews based on the scientific structure, such as historical and archival, lexicographical and bibliographical critical reviews.

Contributions should be in (preferably) Serbian and English, and according to the scientific problem, papers could be written in some other relevant language.

The journal takes into consideration only previously not published papers elsewhere. As this journal has adopted a double blind reviewing policy, editorial board decides whether the article will be published or not and after that the author is informed within the three months. Manuscripts should be send electronically and the appendices (drawings, graphs and schemes) may be send via mail and will not be returned. The editorial board mail and e-mail address are given in the journal imprint page.

Papers for consideration should be prepared according to *Uzdanica* journal's guidelines for authors in order to be reviewed. Papers that have not had all recommended guidelines features will be returned without review to the author.

Manuscript length and font

Articles should be typed in Microsoft Word, 12 pt font, 1.5 spaced. Original scientific and review papers should be approximately 30.000 words in length, reviews and informative contributions approximately 10.000 characters and reports and short notices from 2800 to 3600 characters in length.

Author's name

Full name(s) of the author(s) should be given. It is advisable that middle names' letters are provided as well. Surnames and names of the local authors should be given in the original form preserving the diacritic elements of the alphabet.

Affiliation

The full (official) name of the affiliation in hierarchical order should be given (for example, University in Belgrade, Faculty of Philosophy – Sociology department, Belgrade) and eventually the name of the institution in which the author conducted the presented research.

The name of the affiliation should be placed under the authors' names. Authors' position and title should be excluded.

Contact

The address and the email address should be placed in the footnote at the end of the first page of the article. If there are more authors, only the corresponding author's address should be given here.

Abstract

A concise and factual abstract is required (maximum length from 100 to 250 words). The abstract should state briefly the purpose of the research, the principal results, and major conclusions. An abstract should be placed under the article's title, name(s) of the author(s) but before key words. After the key words, the body of the article should take place.

Summary

If the paper is written in Serbian, the summary should be given in the foreign language. It is recommended that the summary is given in the structured form. The summary length may be 1/10 of the article length and should be placed at the end of the article right after the references.

Key words

The number of key words must not exceed 10. In the article, the key words should be placed after the abstract or the summary.

References

The following APA format guidelines should be applied throughout the paper:

1. Book
In the body of the text: (surname year: page)
In the reference list: surname (year): initial letter of first name and surname, *title*, place: publisher.

Crystal (1999): D. Crystal, *The encyclopaedia of the modern linguistics*, Belgrade: NOLIT.
Chomsky (2008): N. Chomsky, *Hegemony or Survivor*, Novi Sad: Rubikon.
Chomsky (1968): N. Chomsky, *Language and Mind*, Harcourt, Brace and World: New York.

2. Article
In the body of the text: (surname year: page)
In the reference list: surname (year): initial letter of first name and surname, title, *journal title*, issue, place: publisher, pages.

Jovanovic, Simic (2009): J. Jovanovic, R. Simic, Text as the linguistics and communicational structure, *Serbian language*, XIV/1–2, Belgrade: Scientific Society for the cultivation and study of the Serbian language, 325–345.

If several papers from the same author(s) are referenced, then the chronological order of the papers' publication should be followed. If several papers from the same author(s) and from the same year are cited, letters a, b, c etc. should be placed next to the year, for example: 1999a, 1999b…

References with more than one author should be given in the exact order as in the title page.

In the body of the text: (Frankovic, Rakic, Vilotijevic 1973)

In the reference list: Frankovic, Rakic, Vilotijevic (1973): D. Frankovic, B. Rakic, M. Vilotijevic, *Educational work in the boarding schools*, Belgrade: Delta press.

If there are more than three authors, in the text should be cited surname of the first author and “et al.” but in the reference list the names of all the authors should be given in the exact order as in the title page.

Referencing the unpublished papers is not advisable, but if it is necessary, the complete data of the source should be given.

3. Web document

Author’s last name, year, document name (italics), date when the document was retrieved, internet address, for example:

Mercer, S. (2008): Learner Self-beliefs. *ELT Journal* 2008 62(2): 182–183. Retrieved in January 2009 from http://eltj.oxfordjournals.org/cgi/content/full/62/2/182

Drawings, pictures and tables

Pictures (drawings, graphs, schemes) and tables can be prepared by using the computer or the classical technology (wash drawing or pencil and paper). These should be given on the separate file or on the separate papers. The position of tables and figures should be indicated in the manuscript. Tables, pictures and illustrations must be clear and understandable. These should not be paginated and must be numbered by Roman numerals, titles and legends (descriptions, signs, codes and abbreviations). Be sparing in the use of tables and ensure that the data presented in them do not duplicate results already described in graphs.

All figures and images should be black-and-white and of high resolution with a minimum dpi of 300.

Statistical data should be given according to the rules of the relevant scientific methodologies.

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