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THE PROBLEM OF (MIS)UNDERSTANDING THE EQUALS SIGN IN JUNIOR GRADES OF PRIMARY SCHOOL

Abstract. Correct understanding of the equals sign is the key to understanding arithmetic, and a fundamental concept important for learning other areas of mathematics. Research around the world repeatedly mentions problems with correctly understanding the equals sign, emphasising the limited view of the equals sign as a command ”to calculate“ among students. The goal of the research was to identify the development of the concept of equality in line with the operationalised levels (operational, relational, and relational in the context of real-life problem solving) and determine differences in understanding between students of the second (N = 190) and the fourth (N = 210) grade of primary school. The research was carried out using the testing technique. The research results show that students do not possess sufficiently developed relational understanding of the equals sign and that operational understanding prevails. Students of the fourth grade demonstrated better understanding of the equals sign at all levels of understanding than the second graders.

Keywords: equals sign, equivalence, operational understanding, relational understanding, mathematics, mathematics education.

INTRODUCTION

The equals sign – a fundamental concept and symbol in mathematics. It is a concept formed in the earliest days of mathematics education, simultaneously with the formation of the concept of natural numbers. However, “equality is a central – but sorely neglected – concept in mathematics education” (Parslow-Williams, Cockburn 2008: 35). From the aspect of mathematical reasoning, the concept of equality involves at least three components: (a) understanding the equality of two values; (b) understanding the equals sign as the symbol of a relationship; and (c) the idea that there are two sides to the equality (Rittle-Johnson, Alibali 1999). All these components are critical to mathematics and problem solving but are often neglected in mathematics education. It is only when students encounter more abstract

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mathematical content that the need to understand this sign as a symbol representing equivalence arises.

Developing the correct understanding of this concept is often considered an easy task in mathematics education. The equals sign as the symbol of equivalence is a concept that, in a mathematical sense, represents the duality between the concept and the process (Alexandrou-Leonidou, Philippou 2007). If we wanted to define the concept of the equals sign (=) more accurately, we would say that it is a mathematical symbol that expresses equality between variables, constants, or other mathematical expressions. Research studies recognise two basic categories that explain how students use the equals sign, and what said sign essentially represents for them in mathematical equalities:

1) in operational sense, the concept of the equals sign stands for "find the total" or "the solution is";

2) in relational sense, the concept of the equals sign means that two expressions on the opposite sides of this sign express the same quantity/value (Kieran 1981; Knuth et al. 2006; McNeil et al. 2006).

This classification is based on the fundamental perception of the equals sign as an operation or computation on the one hand, and a relationship that expresses equivalence on the other hand. Considering this dual meaning that the concept of equivalence and the sign that expresses it share, it is possible to recognise their different understanding in algebra and arithmetic. A large body of research indicates that the primary source of difficulties that prevent students from correctly understanding the equals sign lies in their previous experience with it (Baroody, Ginsburg 1983; Carpenter, Franke, Levi 2003; Falkner et al. 1999; McNeil 2007, 2008).

Students are first introduced to this symbol in arithmetic classes where the equals sign is used in different forms, which may cause students to acquire an erroneous conceptualisation of it (Kieran 1981). Arithmetic equalities in which the expression is always on the left side of the equals sign often lead students to perceive the equals sign as an instruction that means "calculate" or "find the solution" (Baroody, Ginsburg 1983; Behr, Erlwanger, Nichols 1980; Cobb 1987; Ilić, Zeljić 2017; Kieran 1981, 1989). As a result, students begin to perceive the equals sign as an operation, interpreting it as the command for arithmetic calculations, i.e., they "see '=' as an instruction to complete an operation" (Parslow-Williams, Cockburn 2008: 36).

If we take the equality $12 + 5 =$ ____, for example, the student's first instinct will be to calculate the sum of *12* and *5*. Based on previous experience in arithmetic, students tend to always see the expression and equality in the same way, so that the expected result of the equality above will be *17*. If we consider the student's previous behaviour in encountering the expression and present them with the following equality $12 + 5 =$ ____ $+ 2$, students will struggle to understand the equivalence between the left and the right side of the equality. In this case, they will focus on calculating the sum on the left side of the equality, and, instead of arriving at the correct solution, i.e., number *15*, try to calculate the result of the expression on the left side of the equality $12 + 5 = 17 + 2$, which is incorrect. Through arithmetic content, students acquire the habit of perceiving the left side of the equality as the side where the instructions for the operation are defined, whereas the right side remains exclusively for expressing the results.

Research shows that problems can also be observed in later stages of mathematics education, whereby secondary school students have more difficulties when interpreting the equals sign in "non-standard" expressions (e.g. $3 + 4 = 5 + 2$ and $7 = 7$), than in expressions they are accustomed to (e.g. $3 + 4 = 7$) (McNeil et al. 2010). The roots of this mindset lie in the students' habit to calculate the result of the expression without understanding equality as a whole or identifying relationships between parts of the mathematical expression. Various studies conducted in the USA shows that a staggeringly high percentage of students (about 80%) between the ages of 7 and 11 are unsuccessful in solving problems designed to test their comprehension of mathematical equivalence (Alibali 1999; Baroody, Ginsburg 1983; Cobb 1987; Kieran 1981; McNeil 2007; RittleJohnson, Alibali 1999).

The first arithmetic expressions that students encounter have a huge bearing on the development of their perception and structural understanding of equality and mathematical expressions. Students who correctly understand the equals sign do not view the arithmetic problem as a signal to perform a specific operation, but instead learn to identify the relationship expressed in the equality before calculating the result (Jacobs et al. 2007). Dabić Boričić and Zeljić notice that if expressions are understood "as processes (calculating the value of expressions), and not as objects with a meaning of their own, students will understand algebraic expressions as evaluation procedures, instead of mental entities that can be manipulated" (Dabić Boričić, Zeljić 2021: 31).

The solution to this problem lies in the reshaping of our approach to learning such content. Thus, in situations where students solve problems that involve expressions with addition and subtraction, for example ____ $= 4 + 3$, the equals sign should be replaced with words that indicate equivalence: "is equal to", "two quantities are equal", "something is equivalent to something else", etc. Such examples can help expand the meaning of the equals sign as a concept, shifting it from operational to relational understanding. In the first case, student activity related to mathematical expressions is aimed at calculations, i.e., determining their result/value. Understanding the mathematical expression as an object, on the other hand, refers to understanding its structure as a whole that can exist on its own. Only when the student is able to understand a mathematical expression as an independent object can they reach structural understanding and deeper understanding of the expression, and thus master the concept of equality (Milinković, Maričić, Đokić 2022). Some authors recommend emphasising the link between the different meanings of the equals sign in teaching, especially between the meaning of the symbol, action, and numerical equivalence in order to present numerical equalities in an integrated manner (Molina, Castro, Castro 2009).

The root of all problems with understanding the equals sign lies in the deeper understanding of this concept and developing an understanding of this concept at the relational, instead of just the operational level. In their research, Rittle-Johnson et al. (2011) identified four levels of understanding of the concept of equality where "knowledge levels differ primarily in the types of equations with which students are successful, starting with equations in an operations–equals–answer structure, then incorporating equations with operations on the right or no operations, and finally incorporating equations with operations on both sides" (Rittle-Johnson et al. 2011: 3) (Table 1).

Table 1. Construct Map for Mathematical Equivalence Knowledge (Rittle-Johnson et al. 2011: 3)

Level	Description
Level 4: Comparative relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing that performing the same operations on both sides maintains equivalence. Recognize relational definition of equal sign as the best definition.
Level 3: Basic relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recognize and generate a relational definition of the equal sign.
Level 2: Flexible operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign.
Level 1: Rigid operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.

These levels should not be viewed as separate and unrelated stages, i.e., there is no clear boundary that excludes mutual ties, and students can develop different interpretations of the equals sign at the same time (Jones et al. 2012). This scaled operationalisation greatly facilitates the understanding of the concept of equality.

Based on the considerations regarding the understanding of the equals sign mentioned above and taking into account the research that operationalises levels of understanding of the equals sign (Kieran 1981; Knuth et al. 2006; McNeil et al. 2006; Rittle-Johnson et al. 2011; McAuliffe, Tambara, Simsek 2020), and finally, looking at the outcomes of mathematics education, we can distinguish between four levels of understanding of the equals sign (Table 2).

Table 2. Levels of understanding of the equals sign

Level of understanding	Expected outcomes
Level 4: Real relational	Student understands equivalence in real-world context problems.
Level 3: Complex relational	Student understands equivalence in complex equalities that feature multiple equals signs.
Level 2: Basic relational	Student understands the equals sign as a symbol of equivalence in equalities that feature expressions on both sides of the equality. Student uses relational thinking and understands equivalence in simple equalities.
Level 1: Operational	Student understands the equals sign as a command “to calculate”. Student understands simple equalities that feature expressions on both sides of the equals sign.

In the operationalisation above, the relational level of understanding involves three sublevels: basic relational, complex relational, and actual relational. The lowest sublevel of relational understanding is the understanding of equivalence in situations where we have two sides to the equality (e.g. 3 + 4 = ___ + 2; ___ + 1 = 4 – 3, etc.). Understanding the concept of equality at the complex relational level is demonstrated in situations where the equals sign occurs repeatedly as a link between multiple expressions (e.g. 1 + 3 = ___ + 2 = ____ – 3 = ___). The final sublevel of relational understanding requires the understanding of equality in the context of problem solving. This involves situations in which students are expected to solve specific problems using the balance method, i.e., jumping from one side of the equation to the other.

When it comes to the levels of understanding, it should be emphasised that Kieran (1981) believes that there is a certain misuse of the equals sign among students at all levels of learning, as well as that the operational interpretation of the equals sign begins in the preschool period. The same author argues that certain findings suggest that students’ initial understanding of the equals sign are based on their intuitive understanding of the equals sign as a “do something” symbol, or a symbol indicating where “the answer should go” even before they start formal education. Nevertheless, an intuitive concept formed in this way can be gradually transformed into the relational meaning of the equals sign, which is what teaching aims for, and which would later lay the foundations for learning more abstract content.

For this reason, the main idea behind this research is based on the need to investigate how students understand the equals sign and to examine potential differences in understanding between students of different age in order to identify potential difficulties in the development of this concept in junior primary school.

RESEARCH METHOD

The research goal is to identify the development of the concept of equality in line with the operationalised levels and recognise differences in understanding between students of different ages. Based on the research goal, the following research tasks were defined:

1) Determine the development of the equals sign among students at the operational level;
2) Determine the development of the equals sign among students at the relational level;
3) Determine the development of the equals sign among students at the relational level, in the context of real-world problem solving.

The research sample was selected among the population of students who attended the 2nd and 4th grade in two primary schools in Užice during 2021/2022 (Table 3). The sample was chosen by convenient sampling in order to obtain as objective results as possible. Five classes of second graders (N = 190) and five classes of fourth graders (N = 210) participated in the testing. The reason for choosing second grade students is the fact that the very first knowledge and experience of arithmetic and understanding of the equals sign are acquired in this period, and we wanted to see how firmly that knowledge foundation was built, and which level of understanding they reached. The fourth grade is the final grade in the first cycle of education, so there is a need for a comprehensive understanding of the equals sign as a symbol of mathematical equivalence. In addition, another reason for choosing fourth graders is the fact that similar research by McNeil (McNeil 2007) shows that operational understanding of the equals sign is still most firmly implanted among nine-year-olds.

Table 3. Sample of elementary school students

School	Second grade	Fourth grade	Total
School 1	116 48.33%	124 51.67%	240 100%
School 2	74 46.25%	86 53.75%	160 100%
Total	190 47.5%	210 52.5%	400 100%

The research was implemented using the testing technique. A knowledge test, which aims to determine the development of the equals sign among students, was created for this purpose. The test was created by incorporating models of math problems used by other researchers to illustrate the levels of development of the concept of equivalence (Knuth et al. 2008; Molina, Ambrose 2008; McAuliffe, Tambara, Simsek 2020; Rittle-Johnson, Alibali 1999; Rittle-Johnson, Matthews, Taylor, McEldoon 2011; Cockburn, Littler 2008).

The test was comprised of six problems. Examples of problems are listed in the results section. Students were tested for the duration of one school period and were only allowed to use a pencil to solve the problems. To examine the understanding of the equals sign at the *operational level,* we designed two problems, which aim to determine if students view the equals sign as a symbol of a general idea which translates to “calculate” or “find the solution”. In the first problem, the students had the task to identify the sum that matches the sum 50 + 30, while in the second, they were asked to fill in the blank so that the left and the right side of the equals sign would be equivalent, whereby the expression was located only on one side of the equals sign.

The third and fourth problems involved equalities the solution of which required students to demonstrate that they possessed a developed relational understanding of the equals sign. In order to better examine the development of relational understanding of the equals sign, we distinguish two sublevels: basic and complex relational. The *basic relational level* involved equality-based problems in which operations were located on both sides of the equals sign. The *complex relational level* included equality-based problems with multiple equality signs. In this case, the equivalence involves a sequence of expressions with missing numbers. The fifth and the sixth problem referred to the understanding of the equals sign in the context of real-world problem solving, aiming to examine students’ understanding of the equals sign in real-world problem solving.

Cronbach’s alpha (0.802) indicates good reliability and internal consistency of the instrument used on this sample (Table 4).

Table 4. Cronbach alpha coefficient

Reliability Statistics		
Cronbach’s Alpha	Cronbach’s Alpha Based on Standardized Items	N of Items
0.802	0.797	12

The tests were reviewed by two independent reviewers who have experience in this field, in order to achieve greater objectivity. The level of understanding of the equals sign was determined in relation to success in solving the given problems. The data obtained from conducting the test were processed quantitatively and qualitatively, and given in percentages in the tabular form. A chi-square test was used to test statistical significance of the differences between the variables. The obtained results were also analysed quantitatively, analysing typical errors and incorrect solutions.

RESULTS AND DISCUSSION

UNDERSTANDING THE EQUALS SIGN AT THE OPERATIONAL LEVEL

The first research task aimed to determine the development of the equals sign at the operational level. Looking at Table 5, we can see that both second graders (89.3%) and fourth graders (95.3%) were most successful in solving the problem that required them to calculate and enter the value of the expression: 80 + 20. Interestingly, they were less successful when asked to find the expression with the same value as the one provided (50 + 30). This indicates that students still largely view the equals sign as an instruction to calculate the result. The students were least successful when asked to find the value of the minuend and calculate the correct equality – second graders (66.3%) and fourth graders (79.7%). This shows that students do not view the expression as an independent entity/object, but only as an element to be calculated.

Table 5. Development of the equals sign at operational level

Task	Second grade		Fourth grade		Chi-square
	Successful	Unsuccessful	Successful	Unsuccessful	
Find the sum with the same value as the expression: 50 + 30. a) 50 + 80 b) 80 + 30 c) 40 + 40 d) None of the above	149 79.7%	38 20 3%	190 89.2%	23 10.8%	χ^2 = 6.987, df = 1, p = 0.008
Insert the missing number. ______ = 80 + 20	167 89 3%	20 10.7%	203 95.3%	10 4.7%	χ^2 = 5.168, df = 1, p = 0.018
40 = ______ – 20	124 66 3%	27 12.7%	186 87.3%	27 12.7%	χ^2 = 25.217, df = 1, p = 0.000

If we compare the performance in relation to grade, we can conclude that fourth grade students were more successful than second grade students in every task. The value of the chi-square for each tested problem (Table 5) shows that there are differences in the performance between fourth grade and second grade students, and that they are statistically significant. Such results make perfect sense, especially considering the experience of the fourth graders with more abstract content, which helps them to transcend the operational level. The obtained results are consistent with similar research conducted in different countries (Jones et al. 2012; Knuth et al. 2006; Molina, Ambrose 2008; McAuliffe, Tambara, Simsek 2020; Fyfe et al. 2018; Capraro et al. 2010).

Some of the typical errors that students made when solving these problems are given in Figure 1.

Figure 1. Errors associated with operational level of understanding of the equals sign

In the first example shown in Figure 1, the student completely ignores the minus sign and focuses on the result by changing the sign to get the result that makes more sense to them. The second example of typical errors shown in Figure 1, illustrates the student's tendency to always expect the result on the right side of the equals sign. Similar research indicates that junior primary school students often find equalities, such as *100 = 80 + 20*, to be incorrect (Kieran 1981; Filloy, Rojano 1989; Carpenter, Levi 2000). Research by Knuth et al. (Knuth et al. 2008) reveals that most math textbooks present equalities as operations on the left side of the equals sign, while the right side is reserved for the results of the calculations, which may be one of the reasons why students think the way they do. Booth sees the solution to these problems in the fact that mathematics education requires various modifications of equalities so that the understanding of the equals sign would not be reduced to the expectation that the result of the expression is always located on the right side of it (Booth 1988).

UNDERSTANDING THE EQUALS SIGN AT THE RELATIONAL LEVEL

The second research task aimed to determine the development of the equals sign at the relational level, which comprises two sublevels – basic and complex. The obtained results show that the performance of the second graders in solving these tasks was under 50% (Table 6), and that they only demonstrated partial success in solving the following problem: 40 + 20 + 30 = 40 + ____, achieving 50.8%. Fourth graders were more successful in solving problems that examine the development of their basic relational understanding, except in one example. The problem: 12 + 23 = ____ + 26 turned out to be the biggest obstacle for both student groups, whereby only one in seven second graders managed to solve the problem correctly, and 40.4% of the respondents in the fourth grade. The reason for these results can be found in the operational understanding of the equals sign that is predominant among students. As a result, students put emphasis on the calculation, instead on the equivalence of the expressions on different sides of the equals sign.

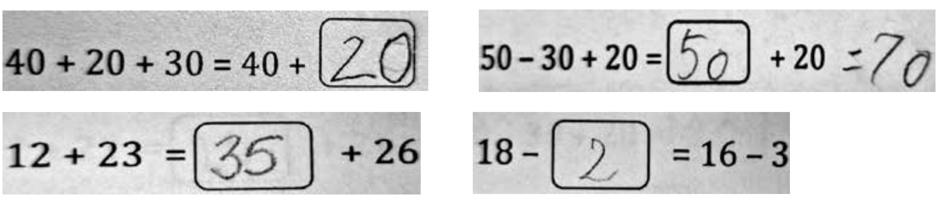
Table 6. Development of the equals sign at basic relational level

Task	Basic relational level				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
Insert the missing number. 40 + 20 + 30 = 40 + ____	95 50.8%	92 49.2%	149 70%	64 30%	χ^2 = 15.351, df = 1, p = 0.000
50 – 30 + 20 = ____ + 20	64 34.2%	123 65 8%	112 52.6%	101 47.4%	χ^2 = 13.619, df = 1, p = 0.000
12 + 23 = _____ + 26	24 12.8%	163 87 2%	86 40.4%	127 59.6%	χ^2 = 37.884, df = 1, p = 0.000
18 – ____ = 16 – 3	51 27.3%	136 72.7%	109 51.2%	240 48.8%	χ^2 = 23.702, df = 1, p = 0.000

If we compare the performance of second graders and fourth graders, we can see that there are statistically significant differences. The value of the chi-square test (Table 6) for each individual task shows that the differences are statistically significant, and that students of the fourth grade are more successful at the basic relational level of understanding of the equals sign.

We will highlight some typical errors that students made when solving these problems (Figure 2).

Figure 2. Errors associated with basic relational level of understanding of the equals sign



The first two examples (Figure 2) show that students ignore the value of the expressions on the left and right side of the equality, and focus on duplicating the expression, while the third and fourth example show operational understanding of the equals sign. It is obvious in these examples that students accept the equals sign as a command to “calculate” the result, thus ignoring the value of the expressions with unknown numbers, i.e., ignoring relational understanding of the equals sign. Similar results and typical errors in understanding of the equals sign have been obtained in other similar research around the world (Duncan 2015; McAuliffe, Tambara, Simsek 2020; Rittle-Johnson et al. 2011).

In addition to the basic relational level, we also wanted to determine students’ understanding of the equals sign in complex situations where the equals sign occurs multiple times. The analysis of the obtained results (Table 7) shows that only 4.8% of second grade students were successful in solving equations with multiple equals signs. Similarly, the percentage of fourth grade students who successfully solved this type of problem was also low (14.6% and 22.1%). Such results show insufficient development of relational understanding of the equals sign as a symbol of equivalence.

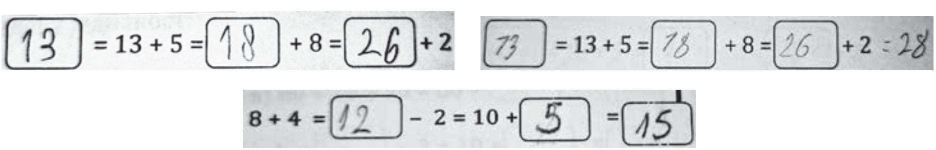
Table 7. Development of the equals sign at the complex relational level

Task	*Complex relational level*				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
Insert the missing number. 8 + 4 = ____ – 2 = 10 + ____ = _____	9 4.8%	178 95.2%	31 14.6%	182 85.4%	χ^2 = 10.499, df = 1, p = 0.001
_____ = 13 + 5 = ____ + 8 = ____+ 2	9 4.8%	178 95.2%	47 22.1%	166 77 9%	χ^2 = 24.618, df = 1, p = 0.000

Despite the fact that both second graders and fourth graders were unsuccessful in solving math problems of this type, if we compare the obtained results, we can see that fourth grade students achieve significantly better results compared to second graders, as confirmed by the values of the chi-square test (Table 7) for each individual problem.

Students made similar typical errors when solving equalities that feature multiple equals signs and require complex relational understanding, as when they solved problems that required basic relational level of understanding (Figure 3).

Figure 3. Errors associated with complex relational level of understanding of the equals sign



The research results show that both age groups of students were least successful in understanding the equals sign at the complex relational level (Table 7). The above examples indicate that students view the equals sign as an instruction “to calculate the result”, i.e., that operational thinking dominates among students when they encounter the equals sign, that the manner in which they perform the operations is largely one-directional, and that they commonly fail to understand the equivalence between the expressions separated by the equals sign. This means that students have not developed relational understanding of the equals sign to a sufficient extent. Students are, thus, unable to highlight the interchangeability of the two sides of the equation (McNeil et al. 2006; Seo, Ginsburg 2003). In order to improve students’ understanding of the equals sign, some researchers suggest to “take care with how you use the ‘=’ sign when demonstrating complex problems with multiple steps. Use arrows if it is necessary to link the successive stages together" (Cockburn, Littler 2008:37).

The fact is that all students have demonstrated significant difficulties in relational understanding of the equals sign, but also that fourth grade students have a more developed relational understanding of the equals sign than second graders. Knuth et al. obtained similar results (Knuth, Stephens, McNeil, Alibali 2006). Their research shows that relational understanding of the equals sign (as a symbol of equivalence) improves over time, as well as that there is a link between the understanding of the equals sign, and the ability to solve equations in later stages of mathematics education. The same authors emphasise the fact that students who have had no prior experience with formal algebra are more successful in understanding and solving equations when older if they possess relational understanding of the equals sign.

UNDERSTANDING THE EQUALS SIGN AT THE RELATIONAL LEVEL IN THE CONTEXT OF REAL-WORLD PROBLEM SOLVING

The third research task referred to the students' performance in understanding the relational level of the equals sign in the context of real-world problem solving. Two problems were selected for this purpose:

1) Two bags of marbles are shown on the picture. The first bag holds 70, while the other holds 30 marbles. How many marbles should change places so that we have the same number of marbles in both bags?

2) There are 28 apples in one basket, and 24 in the other. Nena ate 2 apples from the second basket. How many apples should be transferred from the first to the second basket so as to have the same number of applies in both baskets?

Both problems came with an illustration of the problem situation, so that students would create a clearer picture of the given problem.

The obtained results show that 40.1% of second graders and 64.8% of the fourth graders successfully solved the marble problem (Table 8). When it comes to the apple problem, which is more complex, only a quarter of the second grade students (25.7%) achieved success. On the other hand, almost every other fourth grader (47.9%) successfully solved this type of problem. Compared to the results of the previous research task, we can see that both second graders and fourth graders are more successful in solving the real-world context marble problem in relation to the tasks that require relational understanding of the equals sign in a mathematical context. This fact must be taken into account, especially with regard to the need to improve the students' understanding of the equals sign, and in situations where it is possible to recognise positive aspects of different methodological approaches, such as real-world contexts.

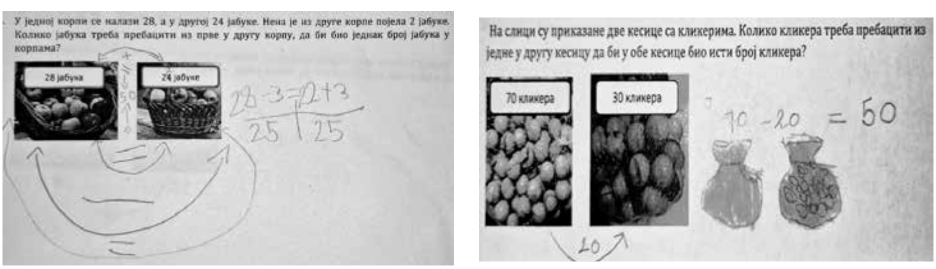
Table 8. Development of relational understanding of the equals sign in real-world problem solving

Tasks	*Relational understanding of the equals sign in real-world problem solving*				Chi-square
	Second grade		Fourth grade		
	Successful	Unsuccessful	Successful	Unsuccessful	
The first problem	75 40.1%	112 59.9%	138 64 8%	75 35 2%	$\chi^2 = 24.368$, df = 1, p = 0.000
The second problem	48 25.7%	139 74 3%	102 47 9%	111 52.1%	$\chi^2 = 20.975$, df = 1, p = 0.000

The value of the chi-test square in both problems shows that there are statistically significant differences between students of the second and the fourth grade. Fourth graders achieved significantly better results compared to the second graders in relational understanding of the equals sign.

Students used different strategies to solve the given tasks (Figure 4). Illustrative examples show the students' need to visualise problems and their use of drawings to facilitate the problem-solving process. Some researchers (Alexandrou-Leonidou, Philippou 2011) believe that the use of multiple visual representations reinforces one's understanding of the equals sign, highlighting the importance of visual and symbolic representations. In the process of solving problems of this type, students relied on different strategies and models that helped make the concept of equivalence more realistic and comprehensible. Visual representations can help in understanding the concept of equality, because they support structural concepts that make abstract ideas more tangible (Fagnant, Vlassis 2013).

Figure 4. Visual strategies for solving real-world context problems



The research by Milinković, Maričić and Đokić (2022) shows that students utilise different forms of visual and schematic representations in solving real-world context problems to present the equivalence of mathematical expressions. According to Dabić Boričić and Zeljić (Dabić Boričić, Zeljić 2021), the key factor of students' success in transforming equivalent expressions lies in developing the meaning of relationships through the process of modelling and other representations. Cockburn & Littler have a similar opinion (2008), arguing that it is necessary to "use concrete apparatus such as balances and visual images to represent a variety of number sentence structures with the 'unknown' on both the left and right-hand sides of the equals sign" (Cockburn, Littler 2008: 37).

CONCLUSION

Taking into account all of the obtained results, it could be argued that fourth grade students achieved significantly better results in the development of all levels of understanding of the equals sign. Despite the fact that the fourth graders were generally more successful, some particulars observable in the obtained results are worth mentioning:

- All students are more successful in operational than in relational understanding of the equals sign;
- Fourth grade students are significantly more successful in relational levels of understanding of the equals sign compared to second grade students;
- Despite being more successful than second grade students, fourth graders nonetheless demonstrate a significant percentage of failure at all levels of understanding;
- Almost one in every ten fourth grade students (except in one example) show that they have not even mastered operational level of understanding to the fullest extent;
- Improved understanding of the equals sign as a symbol of equivalence is evident in older students;
- Evident progress in understanding the equals sign as a symbol of equivalence in real-world context problems.

The research results show that students in junior primary school do not have sufficiently developed relational understanding of the equals sign. A large percentage of students, both in the second and the fourth grade, show that they perceive the equals sign in mathematical equalities as an operation, instead as a relationship that expresses the equivalence of the left and the right side of the equality. Regardless of the fact that there is progress, if the results across the tested classes are compared, the progress is still insufficient to help them understand the equals sign as a symbol of equivalence. The roots of this problem can be found in the fact that the syllabus and curriculum do not pay enough attention to the formation of this concept. There are no clearly defined guidelines or outcomes regarding the development of the concept of the equals sign in the *Rulebook on the Mathematics Syllabus for the First Cycle of Education in the Republic of Serbia* (2019), which only confirms that, despite its importance in elementary mathematics education, not nearly enough attention is paid to this content. Methodologists and practitioners should study the problems accompanying this concept in much more detail and prescribe guidelines that would lead to its proper development.

Considering the obtained results, one of the necessary requirements would certainly refer to introducing changes to the curricula and syllabi, as well as the textbooks, so as to underline the importance of studying this content. The key activity in this process would involve a revision and redesign of the examples and in-class activities, as well as learning examples that serve as the basis for building this concept.

Some of the research (Jacobs et al. 2007) shows that a large number of teachers are unaware of the differences between operational and relational understanding of the equals sign, which is why they tend to disregard the importance of building this concept. For this reason, more attention should be paid to the professional development of teachers through various forms of support, primarily to underline the problems in the correct development of the concept of equivalence and its importance in learning more complex math content. The latter is particularly important given the fact that relational understanding of the equals sign is crucial for the development of algebraic skills, including equation solving and algebraic thinking (Alibali et al. 2007; Jacobs et al. 2007; Kieran 1989; Knuth et al. 2006).

Students encounter various types of equalities from the very first grade, from the simplest arithmetic ones to equations with one unknown, equalities comprising expressions on both sides of the equals sign, etc. Different understandings of the equals sign must be developed simultaneously in teaching, and any operationalisation of the levels of understanding of the equals sign must not result in the interpretation of separate levels as discrete stages (McAuliffe, Tambara, Simsek 2020). In other words, separate levels of understanding are necessary and integral to the development of the concept of the equals sign, but in that process, adequate methods that will speed up the process must be chosen.

Paying insufficient attention to the construction of this concept may lead to undesirable understanding of the concept of equality, which lacks its fundamental property of equivalence. Therefore, it is essential to study children's understanding of this concept and the errors that occur in problem solving, which may result in the subsequent misunderstanding of more complex mathematical content. Some research suggests that students who understand the equals sign as an operational symbol achieve poorer results in algebra in later stages of education compared to those who nurture a relational understanding of the equals sign (Knuth et al. 2006).

Our research focused on equality-based problems of different levels of difficulty, most of which students encounter very seldom in math classes, which only makes the obtained results more valuable and objective. On the other hand, this research is limited due to the fact that the data were obtained through only one written test, so asking additional questions and conducting individual interviews with the students could shed more light on the students' understanding of this concept.

In conclusion, we would like to highlight some of the factors that affect the understanding of the equals sign, as proposed by Molina et al: (a) The cognitive demand of the operations involved in the sentence and, therefore, the students' mastery of arithmetic operations and their number sense; (b) Students' structure sense which includes the capacity to see an arithmetic or algebraic expression as a whole, to split an expression into sub-structures, to detect connections between the structures of different expressions and to recognize in an expression a known structure; (c) Students' knowledge of conventions of mathematic language (Molina, Castro, Castro 2009: 365). In addition to the factors listed above, there is one positive factor that stood out in this research: real-world context, as the basis of the relational understanding of the equals sign. In that sense, real-world context examples are the only ones that can be understood relationally, because the basis of the development of equivalence is found in the real and the tangible.

The research shows that elementary mathematics must focus on the development of relational understanding of the equals sign as one of its primary tasks, because it lays the groundwork for the successful mastering of more complex mathematical content.

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ПРОБЛЕМ (НЕ)РАЗУМЕВАЊА ЗНАКА ЈЕДНАКОСТИ У МЛАЂИМ РАЗРЕДИМА ОСНОВНЕ ШКОЛЕ

Резиме: Правилно разумевање знака једнакости представља кључну основу за разумевање аритметике, али и основни појам важан за учење других области математике. Истраживања широм света наводе проблеме правилног разумевања знака једнакости, при чему се у први план истиче ограничавајући поглед на знак једнакости као наредбу „израчунај". Циљ истраживања био је да се идентификује развијеност појма једнакости према операционализованим нивоима (операциони, релациони и релациони у контексту решавања реалног проблема) и утврде разлике у разумевању између ученика другог (N = 190) и четвртог разреда (N = 210) основне школе. Истраживање је реализовано техником тестирања. Резултати истраживања показују да ученици немају довољно развијено релационо разумевање знака једнакости, већ да доминира његово операционо схватање. Боље разумевање знака једнакости на сваком нивоу разумевања показали су ученици четвртог разреда.

Кључне речи: знак једнакости, еквивалентност, операционо разумевање, релационо разумевање, математика, математичко образовање.