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CHALLENGES OF SOLVING VISUALLY PRESENTED PROBLEMS

Abstract: Organization of the teaching process which enables the acquisition of quality and effective mathematical knowledge applicable in different life situations and which lays the foundation for lifelong learning is based on problem-solving. Teaching how to solve visually presented problems is one objective that contributes to these overall goals. The main research goal of this paper is an experimental examination of the effects of problem-based teaching in the development of mathematical modeling skills involving visually presented problems. The effectiveness of teaching visually presented problems for the development of mathematical modeling skills in solving equations and inequalities in the fourth grade of primary school is examined. The descriptive method was used for the analysis, processing, and interpretation of the research results to investigate the type of errors pupils make when dealing with the visually presented information. Participants in the experimental program showed a higher level of knowledge when solving simple and complex equations and inequalities as well as in composing texts based on given iconic representations of equations and inequalities, as opposed to the pupils not influenced by the experimental model. Given its positive effects on the development of mathematical modeling skills, teaching visually presented problems is justified during the early years of mathematics education.

Keywords: visualization, problem posing, problem-solving, mathematical modeling, equation and inequality.

"We live in a world where information is transmitted mostly in visual wrappings" (Arcavi 2003: 215). Mathematics textbooks for grades K to 4 of elementary school are full of illustrations. Iconic representations are used to present mathematical concepts such as numbers, arithmetic operations, arithmetic laws, etc., often using multiple iconic representations to represent the same concept. Yet, symbolic or language representations are dominant in math problem formulations. School practice shows that mathematical terminology and symbolism tend to represent an obstacle to understanding abstract mathematics concepts or solving math problems (Roubiček 2007). Semiotic analysis can shed light on problems since the problems tend to be caused by different mental representations of the teacher and the pupil's non-conventional writing forms (Ibid.). Visual representations are seen

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as a mediator between pupils' meta-language and mathematics language. Iconic representations, such as drawings, tables, and graphs are used in problem formulations. The question of interest to us is if pupils know how to interpret a visually presented problem.

VISUALIZATION IN PROBLEM POSING

The idea that visual representation is a tool in math reasoning tools is well documented in the literature (Janvier Dufour 1987; Kaput 1987; Cobb et al. 1992; Lesh 1981; Cuoaco, Curcio 2001; Michalewicz, Fogel 2000; Reed Woleck 2001). "Real-world meanings can be acted out, modeled with objects, and drawn with simplified math drawings" (Fuson 2004: 118). "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, to depict and communicate information, thinking about and developing previously unknown ideas and advancing understandings" (Arcavi 2003: 217). According to Norma Presmeg (2006), visualization is taken to include processes of constructing and transforming both visual and mental imagery and all of the inscriptions of a spatial nature that may be involved when doing mathematics. She concentrates on visual constructs as aids in the formation of mental concepts. Arcavi, on the other hand, attends to the role of visualization in mathematical reasoning in problem-solving and problem posing. We consider visualization as a synonym for an iconic representation.

Arcavi believes that seeing things sharpens our understanding and triggers questions that we would not pose otherwise. If we recognize that visualization offers a link between the real world and abstract mathematical concepts it becomes obvious that we should create opportunities for pupils to investigate visually presented problems. If we recognize that visualization offers a link between the real world and abstract mathematical concepts it becomes obvious that we should create opportunities for pupils to investigate visually presented problems. Nicole Venuto and Lynn C. Hart (2017) demonstrated in an investigation the change in the children's ability to explain their mathematical reasoning in writing combined with drawing, suggesting that adding writing to conceptually based lessons improves children's ability to communicate their thinking using visual tools. One way to help pupils become confident in using visualization to problem-solve is to confront them with problems posed in visual forms. Researchers emphasize the importance of problem visualization as a link between external representations and mental images (Singer et al. 2011; Wittmann 2005; Milinkovic, in print). Friedlander and Tabach (2001) argue that using multiple representations of problems provides models which can be used in the problem-solving process. Singer and colleagues (2011) pointed out that the task format underlines the sequence of transfers from external to an internal representation. Barwise and Etchemendy claim that mathematicians value diagrams and other visual tools both for teaching and in the process of mathematical discovery. But despite the obvious importance of visual images in cognition, visual representation remains undervalued in the theory and practice of mathematics. Proofs based on diagrams or graphs are considered to be informal and "half-proved". However, there is a strong push toward the other direction. Nowadays, visual forms of representation are considered to be "legitimate elements of mathematical proofs." (Barwise, Etchemendy, In: Arcavi 2003: 226).

Research shows that pupils have difficulties in dealing with visually possessed problems. Lucia Csachová and Mária Jurečková (2019) found that 10 and 11-year-old pupils undergoing the Slovak nationwide testing have had difficulties linked to their inability to read data from figures. On the other hand, some selected problems were easy for pupils because figures (e.g. illustrations) made the problem-solving process easier (Ibid.). Christian Rütten and Stephanie Weskamp (2019) designed a combinatorial learning environment for fostering reasoning skills: diagrammatic reasoning, conjectures, and justifications in building towers of cubes and cuboids. They stated that diagrammatic reasoning synthesizes "the construction/observation of a diagram, the observation of structural relations among its parts, and the perceptual manipulation and thought-experimentation to infer new possible relations conducive to the attainment of the conceptualization of the Object of the sign-vehicle (the Object-as-it-is)" (Rütten, Weskamp 2019: 364). They point out that the aim of diagrammatic reasoning is the construction of a mathematical argument that warrants the abstract structure of the mathematical object, whereas visualization mediates the emergence of diagrammatic reasoning to arrive at generalization (Ibid.).

Knowledge of representations as was noted earlier is particularly important in problem-solving (Polya 1957; Goldin 1987). Friedlander and Tabach (2001) argue that the teacher's presentation of problem situations with different representations encourages flexibility in pupils' choice of representations. They state that "the presentation of a problem in several representations gives legitimatization to their use in the solution process" (Ibid.: 176). But in practice, teachers rarely consider different representations of problem posing as an important issue.

Sofia Anastasiadou (2009) presents a structural equation model representing the hierarchical structure of translation among representations in frequency concept to 6th-grade Greek pupils. The researcher aimed to contribute to the understanding of the approaches that pupils use in solving tasks related to the frequency concept and to examine which approach is more strongly correlated to their success in such tasks. The structural model that resulted from the analysis of the data confirms the existence of five first-order factors relative to frequency representations. Barbora Divišová and Naďa Stehlíková (2011) report on research that deals with a certain type of geometric problem, i.e., problems effectively solvable without algebraic calculations, for which they observed pupils' preference for (often lengthy) algebraic solutions over (often quick) geometric ones.

VISUALIZATION PROBLEMS IN MATHEMATICS MODELING

Mathematical models, which are results of mathematical modeling, emphasize the structural properties and functional relationships of real-life objects or situations (Lehrer, Schauble 2003, 2007; Lesh, Doerr 2003). "Mathematical model is a formal mathematical record that reflects aspects of the studied phenomenon, often in the form of a graph, equation or algorithm" (Milinković 2014: 46). Also, the mathematical model can be in the form of a diagram (Galbraith, Stillman, according to Barbosa 2006). The model serves as a means of mediating between the real world and the abstract world of mathematics. In other words, models help pupils to solve a problem at every level of abstraction (Milinković 2016). It is used to construct, describe, and interpret certain mathematical situations (Richardson 2004). Terwel and colleagues see in the model "a certain structural form of representation" where representation is a broader and more comprehensive term from cognitive psychology, and model is a term used in mathematics education (Terwel et al. 2009: 27). However, some authors understand the model as a representation of the task that is created and formed, intending to summarize and understand the essence of the tasks (Novick, Bassok 2005).

For mathematical modeling to manifest its positive role in the teaching process, it is necessary to engage teachers in terms of encouraging and developing the following student skills: (1) interpretation of mathematical or scientific phenomena and information presented in the form of text or diagrams; (2) understanding, analyzing, and reading simple examples of tabular data; (3) collection, analysis, and interpretation of data; (4) preparation and compilation of written reports based on the analyzed data; (5) communicating in a group and working together on data; (6) construct models with the group through verbal and written reports (Watters et al. 2000). As it was stated, understanding, interpreting, analyzing, and using diagrams or tabular data are essential in the process of mathematical modeling (Ibid.). The ability to manipulate various representations and use them in a novel problem is considered to be of critical importance in problem-solving (Obradovic, Zeljic 2015).

METHODOLOGY

This paper reports partial results of a larger study investigating the effects of implementing problem-based instruction on pupils. One *objective* of the study was to examine how successful pupils are in composing text tasks related to equations and inequalities based on assigned iconic and symbolic representations, after conducting problem-based teaching. We have determined that the dependent variable was pupils' test scores. The independent variable is the level of mastery of the methods of solving equations and inequalities under the influence of problem-based teaching (model to be introduced).

Research methods. The research is a combination of experimental and descriptive scientific research methods, as well as a theoretical analysis method. We compared the effects of problem-based teaching. Here we particularly explored the elements of research dealing with the development of pupils' ability to understand and use visually wrapped information in the process of problem-solving (Arcavi 2003). The study is an experiment with parallel groups. The experimental program was introduced to each of the two experimental groups, while teaching in the two control groups was carried out in a traditional (classical, established) way. The traditional way of teaching is based on the dominant, lecturing role of the teacher who delivers and transfers ready-made knowledge to the student (Woodlief 2007). Through the frontal form of work with the student, the teacher achieves one-way communication, which disrupts all forms of interaction. In the traditional approach, there is not enough time for the student to engage in independent activities (Bognar, Matijević 2002). *The sample* consisted of pupils of the experimental and control groups, namely 88 pupils who attended the 4th grade of an elementary school in Krupanj and Loznica. The sample has elements of random and cluster sampling. We randomly selected primary schools for the experimental and control groups. An equal number of participants were selected for both groups.

Description of research activities. The research was carried out through multiple steps over 2 months. Here we focus on the results obtained from the initial test and the final test in which we determined the effects of problem-based teaching on the development of mathematical modeling abilities when solving equations and inequalities. With the students of the experimental group (44 students), a total of six lessons related to solving equations and inequalities were realized. In order to check the effects of problem-based teaching in the development of mathematical modeling abilities, when solving equations and inequalities, a testing technique was applied. When conducting the experiment, there were conditions that enabled the presence of researchers in both groups (students of group E attended classes in the morning shift, while students of group K attended classes in the afternoon shift) with the intention of obtaining the most valid results. In both groups, an initial test was conducted first, followed by two final tests, where the first was about solving equations and inequalities, and the second was about understanding the idea of a function. Considering the goal of our paper, we reduced the exposition of the results of the quantitative analysis to a minimum, devoting space to the qualitative analysis of the pupils' answers.

RESULTS

Based on the research results (Tables 1–6) a quantitative analysis was carried out and served as a starting point for performing qualitative analysis.

Table 1. Descriptive characteristics of the sample for all variables

Variables	Number of pupils	AS	S	MIN	MAX
Initial test	88	19.27	4.85	8	30
Final test	88	21.52	10.37	3	38
Complex equations and inequalities	88	3.16	1.73	0	8
Simple equations and inequalities	88	14.47	3.29	6	22
Problem formulation based on iconic and symbolic representations	88	1.63	1.15	0	3

Legend: AS – Arithmetic mean; S – Standard deviation; MIN – minimum result: MAX – maximum result

During the analysis of the initial test, we became familiar with certain subcategories in the tasks with which the pupils of neither the E nor the K groups were able to cope. Looking at Table 2, positive values of asymmetry show that most of the obtained results are to the left of the mean, among smaller values, and negative values of asymmetry show that most of the results are to the right of the mean, among larger values.

Table 2. Elementary statistics for Groups E and K

	Variables	AS	S	MIN
Experimental group (E) N=44	Initial test	19.05	5.14	8
Experimental group (E) N=44	Final test	30.09	5.74	14
Experimental group (E) N=44	Problem formulation based on iconic and symbolic representations	1.68	1.19	0
Control group (K) N=44	Initial test	19.50	4.59	8
Control group (K) N=44	Final test	12.95	5.87	3
Control group (K) N=44	Problem formulation based on iconic and symbolic representations	1.57	1.11	0

In the final test, tasks f1a3 and f1b3 are parallel versions of task i2a3 that the pupils solved in the initial test and were related to solving simple equations where it was necessary to present the given equations in an iconic way. Table 3 indicates that the implementation of the introduced Model (Experimental Program) was of great importance for the success of pupils in the experimental group.

Table 3. Performance of E and K groups during the iconic representation of simple equations on the Final Test

Iconic representation of simple equations (h – 430 = 2350, 560 : h = 70)			
	N	f $1a_3$	f $1b_3$
E group	44	97.72%	56.82%
K Group	44	4.54%	0%

The E group was significantly more successful than the K group, which demonstrates the fact that the pupils, with the help of the introduced Model, managed to overcome the difficulties they encountered in the initial test in a task with the same requirements. We think that the low performance of pupils of the K group stems from the fact that the pupils simply did not encounter the iconic representations of equations, but also that they have not developed enough mathematical modeling ability. Also, one of the tasks that turned out to be difficult for pupils of both groups on the initial test was solving an inequality with the help of a table. We will see what the situation is after the introduced Model in Table 4.

Table 4. The success of pupils of E and K groups in solving the inequation using the table

Solving inequality using a table (25 · h < 200)		
	N	f $2a_3$
E group	44	97.72%
K Group	44	0%

Again, we are faced with the fact that the E group showed great interest in the Model that was introduced and thus demonstrated its high success in overcoming the problem it encountered in the initial test. On the other hand, the pupils of the K group simply ignored the table and solved the given inequality in the usual way using knowledge related to the expression of unknown components.

Table 5. Achievement of pupils of groups E and K in transposing a simple inequality from iconic to symbolic form

Transforming an iconic representation of inequality into a symbolic one		
	N	f $4a_1$
E group	44	97.72%
K Group	44	9.09%

The percentage difference between the control and experimental groups is really large, which again confirms the high efficiency of problem-based teaching in the development of mathematical modeling abilities. When solving tasks related to complex equations, we previously talked about the fact that the pupils of the

E and K groups had certain difficulties during their iconic presentation, then we introduced the Model to the experimental group to find out if the situation at that time would change. In the following text, we will see whether and to what extent the pupils developed the ability to do mathematical modeling and how successfully they were able to transpose equations from the iconic frame to the symbolic one.

Table 6. Success of pupils of groups E and K in transposing complex equations from iconic to symbolic form

Transforming an iconic representation of a complex equation into a symbolic one					
	N	f6 and$_1$	f9a$_1$	f10a$_1$	f10a$_3$
E group	44	95.45%	45.45%	77.27%	88.64%
K Group	44	45.45%	0%	6.82%	2.27%

Note that a few pupils from the E group simply completed the picture very easily, and based on the picture, they immediately came to the correct solution without setting the equation. They ignored the request for the symbolic notation of the equation; however, the final answer and solution were correct. The situation with the group (K), which was not influenced by the Model, was very different from the E group, because only 3 pupils correctly wrote the symbolic notation of the equation, and only one pupil completed the given picture. In the K group, an additional 2 pupils correctly presented the equation symbolically based on the realistic situation, ignoring the picture completely, and thus came to its solution.

To analyze the types of difficulties that pupils had when composing tasks based on visual or symbolic representations, we created subcategories: (a) representation of a symbolic record with an adequate iconic representation; (b) solving the inequality with the help of a table; (c) composing the appropriate text based on iconic and symbolic representations and (d) noticing the rules based on which a certain table was filled in, which was a reflection of the idea of the function. To confirm the above, with the help of the following Table 7, the success of solving tasks by pupils according to the specified subcategories can be seen.

Table 7. Percentage of pupils who successfully solved the tasks according to the specified subcategories on the initial test

		Transforming symbolic notation into an iconic representation		Solving the inequality using a table		Problem posing based on iconic and symbolic representations	Drawing conclusions (functional idea)
	N	i2a$_3$	i10a$_3$	and 5a$_3$	and 5b$_3$	i6a$_5$	i11a $_6$
E group	44	18.18%	9.09%	29.55%	22.73%	31.82%	2.27%
K Group	44	13.64%	6.82%	20.45%	15.91%	22.73%	2.27%

After the initial measurement, the experimental program was introduced to the experimental group, after which the first final measurement was carried out.

Table 8. Final test results

Competences in Mathematical modeling	Problem posing based on iconic and symbolic representations		Total (10 tasks)
Task	4.	6.	
E group	130	106	
	236		1366
K Group	11	37	
	48		569

Differences in the achieved success between the experimental and control groups at the final measurement are evident, where the experimental group shows a marked improvement both in relation to the initial measurement and in relation to the control group in the final measurement, while the control group shows a certain decline considering its initial state and a drastic drop compared to the group in which the experimental program was introduced (Table 8).

DISCUSSION

During the analysis and interpretation of the results of the initial test, one of the main difficulties faced by the pupils when solving the tasks was related to the composition of the relevant text with the given iconic and symbolic representations of the equations. In the final test, the pupils had this requirement in two tasks: task f4a5, where it was necessary to first compose the inequality we talked about earlier in symbolic form, based on the iconic representation of the inequality, and then come up with a text that describes it, and task f6a5, where it was necessary to compose the symbolic form of the equation based on the iconic representation and then compose the text task that corresponds to the given image (equation). Considering the results, we see that the E group achieved significantly better results compared to the control group in both cases. We believe that in this case as well, the Model that we applied to the E group stood out and proved to be very effective.

Mistakes made by pupils when solving tasks on knowledge tests, i.e. when emphasizing the degree of development of mathematical modeling skills, can be categorized according to the following groups:

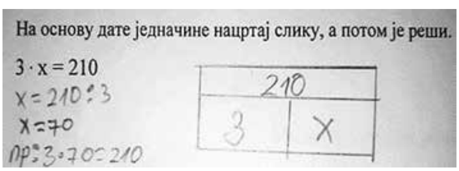
- neglecting the structure of the expression (equation, inequality) in its iconic presentation;
- wrong transposition of the equation from the iconic to the symbolic environment;

- ignoring the iconic and symbolic form of expression (equations and inequalities) when designing the math task in textual form;
- misunderstanding the structure of the task textual formulation and mixing formulas;
- inadequate execution of rules and conclusions based on the visual display of the problem.

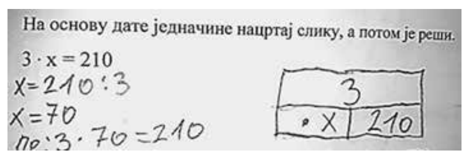
We will support each of the mentioned types of errors with adequate examples (pupils' work) that will complete our analysis. Given that the pupils solved the simple equation ($3 \cdot h = 210$) very easily, the second task on the initial test gave them a lot of difficulties in the sense that it was necessary to convert the same equation from a symbolic form to an iconic. Many pupils neglected the structure of the set equation and were guided by their previous experience on the first task in which they also very easily supplemented the set iconic representations of the equations when it was necessary to sketch the picture corresponding to the equation themselves, errors occurred. How pupils represented an equation of the form $3 \cdot x = 210$ is illustrated in Example 1.

Example 1. Misunderstanding the structure of an expression (equation) when designing an iconic representation

1a) P.24E-i2 (P – pupil)



1b) P.81K-i2

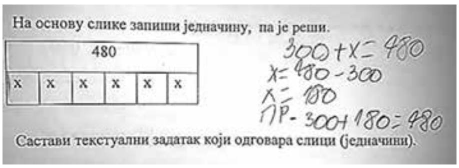


As the pictures show, the pupils successfully came to the solution of the given equation, but the iconic representation is not relevant to its symbolic record. We believe that the pupils did not pay attention to the structure of the equation and instead were guided by the equations related to addition and subtraction and their pictorial forms, thus coming to the wrong iconic representation of the simple equation $3 \cdot x = 210$. In both examples (Example 2a and b) it can be seen that the pupils need to include the given symbolic components of the equation in some way only in the table that they have previously sketched, without taking into account what the structure of the set of equations represents. This form of error appeared very often in the works of pupils of both E and K groups, and as such, we classified it in a special category of the most common mistakes made by pupils when solving tests.

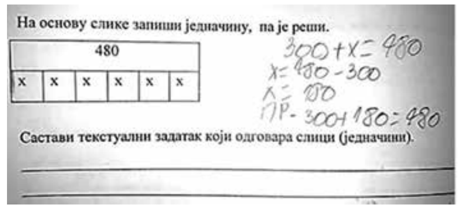
The next mistake refers to the situation in which the pupils underwent the task that examines the ability to transpose information from an iconic to a symbolic environment (pupils should represent the given equation symbolically) as an illustration that "beautifies" it. Symbolic representations of a given iconic representation are not relevant to it (Example 2).

Example 2. Misunderstanding the iconic form of the equation

2a) P.5E-i6



2b) P.70K-i6

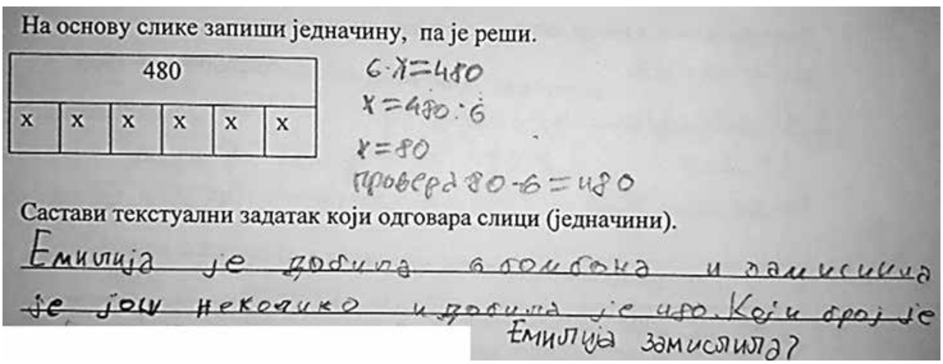


The activity of transposing information from an iconic to a symbolic environment caused a lot of difficulties for the pupils, and in connection with that, during the analysis of the papers, we came across three subcategories of this problem. Observing the work under a) P.35E-i6, it is seen that the student first does not understand the iconic structure of the equation of the form $6 \cdot x = 480$, and then writes down the symbolic structure of a mathematical expression that is meaningless ($x = 480 + x = 6$). In the second paper under b) P.5E-i6, the student shows that he somewhat understands the structure of the picture, but most likely due to carelessness when reading the instructions for solving the task, he writes down an inequality instead of an equation.

We see that this pupil unsuccessfully tried to compose the text (which will be discussed later), firstly because he did not write down an adequate equation based on the picture, and then due to carelessness when reading the second request, the student writes down a sentence that also sounds meaningless instead of a text task. The third paper, under c), also shows the pupil's misunderstanding of the iconic representation of the equation ($6 \cdot x = 480$) in which the student constructs the equation by adding elements that the equation does not contain in its iconic form. For unknown reasons, the student uses the number 300 as one of the components of an inadequately written equation and sees the set iconic representation as an equation of the form $y + x = z$, instead of $y \cdot x = z$. After all the above, we can conclude that the transposition of the equation from the iconic to the symbolic environment was very difficult for the pupils of the fourth grade of elementary school. However, the pupils may have encountered difficulties in this task because they did not have any or sufficient experience in modeling and symbolic representation of equations. The data we obtained after the introduction of the experimental program support the previously stated (pupils do not have adequate experience in modeling) because the pupils (E group) who were influenced by the Model showed that they overcame this type of problem and thus achieved a much better result than the control group (which was discussed earlier). We have grouped all the mentioned examples under one category, i.e., the wrong transposition of the equation from the iconic to the symbolic environment, because we believe that every type of problem mentioned is based on a misunderstanding of the iconic form of the equation. The next problem faced by the pupils is related to the situation in which the pupils ignore the iconic/ symbolic form of the equation, and because of that, they were not able to come up with an adequate text task that would complete the given equation. In Example 4, we present work from a student to elucidate the previously mentioned difficulty.

Example 3. Ignoring the iconic/symbolic form of the equation when designing a word problem

3a) P.7E-i6

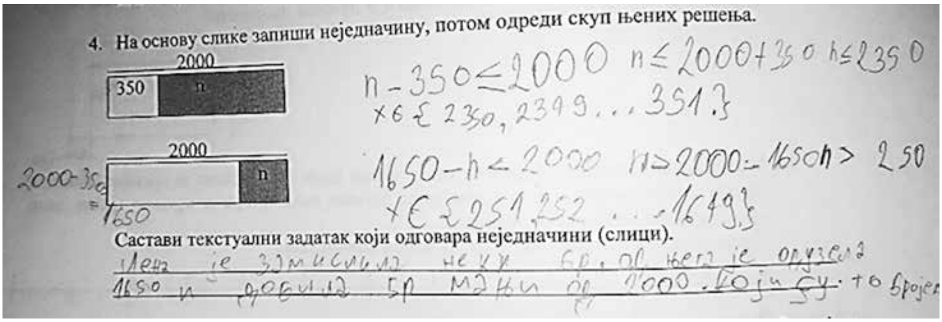


The student's solution shows that the student of the experimental group knows and understands the iconic representation of the equation, then correctly represents its symbolic notation, and arrives at the correct solution of the equation. However, the problem arose when it was necessary to compose an adequate text corresponding to the given equation. In this case, the student composes a task that is meaningless both in the textual and mathematical context. First, there was talk about "imaginary candies", while at the end of the tasks, those candies were reduced to "imaginary numbers". The student started the text with one idea and ended with another idea.

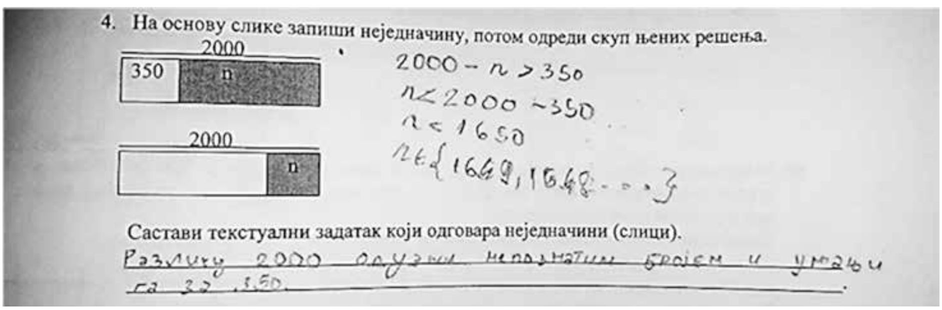
One gets the impression that the student paid attention to the symbolic representation of the equation because he mentioned the components of the equation in the text, but still during the process of writing the text he "lost his train of thought" and thus made the problem formulation meaningless. The problem that we classified in the same category as the previous one was related to composing a text task based on an iconic/symbolic representation, but this time with inequations.

Example 4. Ignoring the iconic/symbolic form of inequation when designing a word problem

4a) P.60K-f4



4b) P.27E-f4

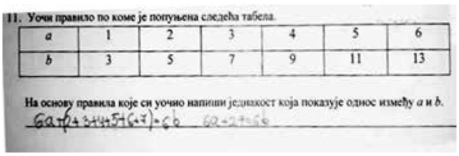


Example 4a (P.60K-f4) is the work of a student of the control group, based on which we can conclude that the student does not understand the iconic form of the given inequality, which creates the conditions for the further emergence of problems when composing the text task that was supposed to describe that inequality. Example 4b (P.27E-f4) shows that the student of the experimental group understands the iconic form of the inequality, writes it down correctly, solves it, and gives a set of solutions. In this example, the student shows that he understands the connection between the created pictorial representation of the inequality and its symbolic notation, which is not the case in Example 4a. However, the problem of composing the text was manifested again. It is possible that these results were obtained because the pupils' previous experiences related to such requests did not occur.

The last mistake, or rather the difficulty that the pupils encountered, is related to a thematic area that was outside the area of equations and inequalities. It was about the idea of function.

Example 5. Inadequate determination of the rules and dependencies between the sizes based on which the given table was filled

5a) P.62K-i11



11. Уочи правило по коме је попуњена следећа табела.

a	1	2	3	4	5	6
b	3	5	7	9	11	13

На основу правила које си уочио напиши једнакост која показује однос између *a* и *b*.

5b) P.14E-i11

11. Уочи правило по коме је попуњена следећа табела.

a	1	2	3	4	5	6
b	3	5	7	9	11	13

На основу правила које си уочио напиши једнакост која показује однос између *a* и *b*.

In both examples of this type of task, the pupils clearly show that they do not understand how to derive a rule based on the given data from the table, that is, the dependence between the variables a and b. They show that in their previous experience they have no traces of the idea of a function. This was the task on the initial test that was performed most unsuccessfully by the pupils of both groups.

CONCLUSIONS

In this paper, we addressed the role of mathematical visualization in problem posing. We have argued that mathematical visualization provides cognitive accessibility to problems. How useful visualization may be in mediating the pupils' mental passage from a realistic world to an abstract one as needed was investigated in the reported empirical study. Based on the evidence it was argued that visualization is an important pedagogical tool in mathematics teaching, as it provides a modality of reasoning. We also recognize that the evidence of research in model situations is limited the importance of the context of learning and teachers' expertise for student outcomes. We need to make an effort to strengthen pupils' ability to understand and deal with visually presented information. Teachers need to learn about the most common mistakes, doubts, and difficulties of pupils when dealing with visually presented information to have a realistic idea of student possibilities and achievements. We believe that this paper contributes to that. Difficulties translated into mistakes made by students when solving equations and equalities were classified as the following: (a) Misunderstanding of the structure of expressions (equations) when designing an iconic representation (Example 1); (b) Misunderstanding the iconic form of the equation (Example 2); (c) Ignoring the iconic/symbolic form of the equation when designing the text task (Example 3); (d) Ignoring the iconic/symbolic form of inequality when designing a text task (Example 4); (e) Inadequate determination of the rules and dependencies between the sizes based on which the given table was filled (Example 5). Namely, after the implementation of the experimental program, the results of this research showed that the students who learned content related to equations and inequalities through the implementation of problem-based teaching showed significantly fewer errors on the final knowledge test compared to the initial test, while the students of the control group made the same mistakes on the final test. Some of the requirements that modern educational practice puts before us can be realized through a series of recommendations that arise from our research: 1) the teaching process should be enriched with innovative models such as problem-based teaching; 2) continuously guide students to be as active participants in the teaching process as possible in terms of encouraging them to independently draw conclusions, approach problems from different points of view, model, exchange experiences and knowledge with other students, etc.; 3) students should be trained to connect different problem situations with the real world, so that they can more easily understand their abstraction; 4) pay more attention in the lessons in the field of equations and inequalities to the connection between iconic and symbolic representations in order to develop the ability of mathematical modeling. The importance of connecting problem situations with their real context was also demonstrated during the research conducted by Lazić and Milinković (2017). In their research, it was shown that students are very happy to approach solving problems related to fractions given in the form of pictures, tables, and graphs, while the symbolic presentation of fractions without an accompanying picture caused difficulties, which indicates a similarity with our results in this paper. Milinković, Mihajlović, and Dejić (2019) also conducted research that showed that students aged 11 can very successfully use different representations and models when solving mathematical problems. Namely, if students are properly trained to correctly connect iconic and symbolic representations, they will not have difficulties when transforming one into the other, which was the result of research in our E group after the introduction of the experimental program. By summarizing the entire analysis and interpretation of the results, we concluded that the implementation of problem-based teaching in mathematics classes effectively aids in the development of mathematical modeling abilities, particularly the development of competencies in dealing with visually presented data.

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ИЗАЗОВИ РЕШАВАЊА ВИЗУЕЛНО ПРЕДСТАВЉЕНИХ МАТЕМАТИЧКИХ ПРОБЛЕМА

Резиме: Организација наставног процеса која омогућава стицање квалитетних и ефективних математичких знања примењивих у различитим животним ситуацијама и која поставља основу за целоживотно учење заснива се на решавању проблема. Подучавање у решавању визуелно представљених проблема један је од циљева који доприноси овим општим циљевима. Основни циљ истраживања представљеног у овом раду јесте експериментално испитивање ефеката проблемске наставе у развоју способности математичког моделовања које укључује визуелно представљене проблеме. Испитује се ефикасност наставе визуелно представљених задатака у развијању вештина математичког моделовања у решавању једначина и неједначина у четвртом разреду основне школе. Дескриптивна метода коришћена је за анализу, обраду и интерпретацију резултата истраживања како би се испитале врсте грешака које ученици праве при раду са визуелно представљеним информацијама. Учесници експерименталног програма показали су виши ниво знања у решавању једноставних и сложених једначина и неједначина, као и у састављању текстова на основу задатих иконичких приказа једначина и неједначина, за разлику од ученика који нису били под утицајем експерименталног модела. С обзиром на позитивне ефекте на развој вештина математичког моделовања, визуелно представљена проблемска настава оправдава потребу за применом у настави математике на млађем школском узрасту.

Кључне речи: визуелизација, постављање проблема, решавање проблема, математичко моделовање, једначина и неједначина.