Boris Čulina
University of Applied Sciences Velika Gorica
Department of Mathematics

UDC 51:373.29-053.4
DOI 10.46793/Uzdanica19.S.199C
Original research paper
Received: August 31, 2022
Accepted: November 25, 2022

# WHAT IS MATHEMATICS FOR THE YOUNGEST?

(What an old mathematician learned about mathematics from his granddaughter Nina)

*Abstract*: While there are satisfactory answers to the question "How should we teach children mathematics?", there are no satisfactory answers to the question "What mathematics should we teach children?". This paper provides an answer to the last question for preschool children (early childhood), although the answer is also applicable to older children. This answer, together with an appropriate methodology on how to teach mathematics, gives a clear conception of the place of mathematics in the children's world and our role in helping children develop their mathematical abilities. Briefly, children's mathematics consists of the world of children's internal activities that they eventually purposefully organize in order to understand and control the outside world and organize their overall activities in it. We need to support a child in mathematical activities that she does spontaneously and in which she shows interest, and we need to teach her mathematics that she is interested in developing through these activities. In doing so, we must be fully aware that the child's mathematics is part of the child's world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it. From the point of view of this conception, the standards established today are limiting and too focused on numbers and geometric figures: these topics are too prominent and elaborated, and other mathematical contents are subordinated to them. Adhering to the standards, we drastically limit the mathematics of the child's world, hamper the correct mathematical development of a child, and we can turn her away from mathematics.

*Keywords*: preschool mathematics, standards for preschool mathematics, the NCTM standards, the "new mathematics" movement.

*Words of caution*: My four-year-old granddaughter Nina has been my main motivation and a "collaborator" for the views expressed here. I wrote the views in the deep conviction that they can enable a better mathematical development of children than the established standards, and that as such they are worth sharing. For definiteness, I chose the NCTM standards (NCTM 2000), a very clear and precise document with a lot of value but, in my opinion, limited and improperly balanced content, published by the National Council of Teachers of Mathemat-

boris.culina@vvg.hr 

ics – the leading organization of mathematics teachers in the USA and Canada. I will primarily refer to Chapter 4: Standards for Grades Pre-K-2. As far as I know, nothing substantial would have changed in further considerations had I taken some other standards for a reference. I believe that what many teachers and educators do or want to do is in accordance with the conception presented here. However, I am a mathematician with expertise in mathematical logic and the foundations of mathematics, and many years of experience in teaching higher mathematics. I have neither the wider experience nor the expertise in the field of mathematics education of the youngest. If we add that thinking about the mathematics education of the youngest is a sensitive topic where wrong attitudes can have significant consequences, it is inevitable to conclude that the views expressed in this article should be subjected to intensified criticism. Given that I am not an expert in the field, my knowledge of the literature and various theories of children's (mathematical) education is far from systematic. I searched the literature as much as I needed to draw conclusions about the problems that interested me. Such an approach led to a non-systematic use of the literature and a non-systematic connection of the conclusions presented here with the relevant literature. My initial guide was the book (Servais, Varga 1971) that I read a long time ago and which left a deep impression on me, especially Varga's introductory article. His words "Every child, by nature, likes learning just as he likes eating" (page 28) were vividly engraved in me. Most of the students I worked with no longer had that hunger for learning mathematics. For too many of them, this hunger for learning has been replaced by an aversion to mathematics. I have always considered it an unacceptable state of affairs. However, when I entered the world of mathematics learning for the youngest and realized that such a situation exists there, moreover, that it arises there, I experienced it as violence against children. What especially bothers me is hearing that a child is not good at math. In addition to the fact that we should be very careful with such claims, how can we even claim this if we do not properly understand what mathematics is? Rigid standards lead to such unnecessary disqualification of children. I am deeply convinced that changes in the mathematics education of the youngest are necessary and that the time for the changes has come. A sufficiently broad understanding of mathematics is very important here. I hope this article will contribute to such an understanding.

## 1. INTRODUCTION

Assisting Nina in her mathematical development, I realized that this development is very important for her overall development and that I understand quite well how to teach her mathematics, but, to my surprise, I do not know what mathematics to teach her, although I've been doing math my whole life. I started searching the scientific literature. There I found confirmation of the almost crucial importance of mathematical development at the preschool period for the future mathematical and overall development of a child. For example, we can read (Moss et al. 2016: 154): "Accumulating evidence confirms that children's mathematics learning in the first six years of life has profound, long-lasting outcomes for students in their later years – not only in relation to their future mathematics achievement but also in terms of overall academic success." These studies only confirm for mathematics educators what the creators of early childhood education realized long ago: that the first six years of life are the most important period in a person's development.[1] Furthermore, recent research has shown that children at this age are much more mathematically capable than previously thought. Thus, in (English, Mulligan 2013) editors begin the preface (page 1) with these words: "This edited volume emanated primarily from our concern that the mathematical capabilities of young children continue to receive inadequate attention in both the research and instructional arenas. Our research over many years has revealed that young children have sophisticated mathematical minds and a natural eagerness to engage in a range of mathematical activities. As the chapters in this book attest, current research is showing that young children are developing complex mathematical knowledge and abstract reasoning a good deal earlier than previously thought." Regardless of my experience with Nina, these results did not surprise me at all. It is known that early childhood is a period of exceptional creativity and imagination[2] (if appropriate conditions are ensured for the child), and in the views of mathematics that will be presented below, creativity and imagination are the key elements of mathematical activities, although in general culture these abilities are usually associated with art. I also found out that my teaching of Nina was in accordance with a certain methodology of mathematical teaching of children. This methodology is mostly established and provides satisfactory answers to the question of how to teach children mathematics. In short, *the child's mathematical activities must be part of the child's world – part of her daily activities, part of her play, incorporated into children's stories that she enjoys listening to. Mathematical activities must have their motivation, meaning and value in the child's world, and not from the outside, in the world of adults. In developing mathematical abilities, children must have freedom and not the pressure to achieve pre-established learning outcomes.* I have singled out two of the many quotations that confirm this methodological approach. Tamás Varga (Servais, Varga 1971: 16) writes: "To realize and enjoy the beauty of mathematics, pupils must be given sufficient opportunity for free, playful, creative activity, where each can bring out his own measure of wit, taste, fantasy, and display thereby his personality." Susan Sperry Smith (Smith 2001: 16) writes: "Most experts believe that children's play is the key to mental growth. Time to play and a wide variety of

[1] It was this knowledge that motivated Friedrich Fröbel to design and introduce kindergartens into modern society in the first half of the 19th century.

[2] See, for example, the chapter on creativity in Bilbao (2020).

concrete materials are essential. Children should not be rushed to finish a project or hurried from one activity to another." As Georg Cantor said that the essence of mathematics is in its freedom (Cantor 1883: 19), we could also say that the essence of mathematics education of a child is in her freedom. It is up to us to help to guide children in developing their mathematical abilities, respecting their world and their individuality – which activities and at what stage of his growth attract him – and providing a social environment for free communication and joint action. At the community level, this requires developing the awareness of the importance of children's mathematical development and the willingness of the community to invest money in creating adequate working conditions for educators and teachers. Developing such awareness is especially important because the existing school systems, as far as I know, are generally contrary to this methodology, in theory with their uniformity and evaluation system, and in practice with challenging working conditions for educators and teachers. This methodology is not only related to mathematics education but refers to the overall education of children. Although its roots can already be found in ancient Greece[3], this methodology was developed in the 19th century by the founders of modern education, Pestalozzi, Fröbel, Montessori and many others (see, for example Lascarides, Hinitz 2000).[4]

But what about the question "What math should we teach children?" I was not satisfied with the answers I found. Numbers and geometry? That answer could have been satisfactory until the middle of the twentieth century. Truly, until the middle of the nineteenth century mathematics was described as the science of numbers and (Euclidean) space. The appearance of non-Euclidean geometries which are incompatible with Euclidean geometries but are equally logical in thinking and equally good candidates for the "true" geometry of the world has definitely separated mathematics from the truths about nature. This separation has freed the human mathematical powers, and it has caused the blossoming of modern mathematics. The new views of mathematics have spread into the mathematics community mainly through the works of Richard Dedekind, David Hilbert, Emmy Noether, Van der Waerden and Bourbaki group, and they have become the trademark of modern mathematics. With the end of World War II, it became clear that there was a big discrepancy between modern mathematics which proved to be very important for modern society and mathematics taught in school. The "new mathematics" movement of the 1950s and 1960s, which was the most intense in USA, tried to introduce modern mathematics to school. This movement unfortunately failed, not only because of social circumstances but also because of the one-sided structural-

[3] For example, in Lascarides, Hinitz (2000: 9) we can find: "The Greek idea of childhood is interwoven with play. The Greek word for child is *pais*, and the word for *I play* is *paizo*, both having the same root."

[4] Studying their works, I was personally fascinated by the wealth of educational knowledge they left behind and frustrated by the ignorance of this knowledge in today's wider educational practice.

ist view of mathematics inspired by the Bourbaki group.[5] Thus, for example, the famous American mathematician Marshall Stone, in an article (Stone 1961) in which he very clearly explains the changes that have occurred in mathematics, characterizes modern mathematics "as the study of systems comprising certain abstract elements and certain abstract relations prescribed among them". Stone believes that this must be the backbone of mathematics education. At a symposium organized by The Society for Industrial and Applied Mathematics (Carrier et al. 1962), another famous mathematician Richard Courant clearly identified the dangers of such an approach: "The danger of enthusiastic abstractionism is compounded by the fact that this fashion does not at all advocate nonsense, but merely promotes a half truth. One-sided half-truths must not be allowed to sweep aside the vital aspects of the balanced whole truth." Just as half a dinghy is no longer a dinghy, so half the truth about mathematics is not the truth about mathematics. Unfortunately, we will never know if the reform would have been successful had its creators complemented their program with the "other half" of the truth about mathematics, which includes its content and usability, as well as its origin and development. The reform took place in such a way that the younger the age, the worse the results became. Although the reform failed, its traces remained in modern mathematics education, even of the youngest. For example, although the NCTM standards are dominated by numbers and geometry, there are many structural elements in the elaboration of these themes that were highlighted by the creators of the "new mathematics". Also, additional contents are included: classifying (sets), sorting (equivalence relations), ordering (ordering relations), matching (functions), patterns, chance, change, etc. However, they are mostly subordinated to the numbers and geometry of figures. Even if we single out these contents in relation to numbers and geometry, my feeling was that they still offer a too limited answer to the question of what mathematics to teach children. Thinking about this question, I realized that it is closely related to the question "What is mathematics?" – a question that I have been dealing with all my life. Having thus connected what I was doing in mathematics with the problem of what mathematics to teach Nina, I began to unwind the knot.

In the next section, I briefly describe the philosophy of mathematics that I stand for. In the third section, I present the answer that this philosophy of mathematics gives to the question of what mathematics to teach children, and I compare that answer with the established standards of mathematics education. In the remaining two sections, I highlight some elements that I believe are particularly important in the mathematical development of the youngest and give some comments on the NCTM standards.

[5] A detailed analysis of the „new mathematics" movement can be found in Phillips (2015).

## 2. WHAT IS MATHEMATICS?

The philosophy of mathematics has not yet given a generally accepted, unambiguous and well-developed answer to the question "What is mathematics?". Fortunately, mathematics survives quite well without a definitive answer to this question, although the philosophies of mathematics have strongly influenced the development of mathematics. The field of mathematics education is also developing regardless of the lack of a definite answer to the question of what mathematics is. Yet, the answer necessarily affects mathematics education, as do various psychological views on the nature of child development. One should be very careful because wrong or one-sided answers can have negative consequences, as the example of the "new mathematics" movement has shown. Roughly, philosophical answers to the question of what mathematics is can be divided into two groups. According to one group we discover mathematics, according to another we create mathematics. Simply put, the various philosophies of mathematics are divided according to whether natural numbers were discovered or created. Which view we adopt should certainly have an impact on how we teach numbers to children. For example, if numbers exist in a particular world of Plato, then special methods need to be devised to get children into that world and teach them how to discover numbers there. If numbers are created, then we need to show children how to create them. The philosophy of mathematics I stand for has nothing in common with realistic views of mathematics, according to which mathematical objects and mathematical worlds belong to the external world. According to this philosophy of mathematics, the human being and the human community create mathematics, just as they create, for example, works of art. This view of mathematics is close to Hersh's humanistic philosophy of mathematics (Hersh 1997) and Ernest's social constructivist philosophy of mathematics (Ernest 1997) and can be considered a certain elaboration and modification of their views in one part. This philosophy encompasses structuralism, constructivism, formalism, and fictionalism in a way that avoids their one-sidedness. It is described in detail in (Čulina 2020). Here I will briefly present it and draw the consequences for the mathematical upbringing of children. As far as I can see, the only source of its one-sidedness may be in not accepting mathematics as part of reality. From my personal teaching experience, I know that looking at mathematics as a free and creative human activity is a far better basis for learning mathematics than looking at it as an eternal truth about some elusive world.

The philosophy I will briefly present here has the same roots as modern mathematics – in the emergence of non-Euclidean geometries that led to the separation of mathematics from truths about reality. According to this philosophy, mathematics is not a science of the truths of the world, but it is a means of discovering those truths; it is human invention whose purpose is to be a tool of our rational cognition and rational activities in general. This purpose significantly influences its design and determines its value. Dedekind summed it up nicely with the example of numbers (Dedekind 1888): "[…] numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things". Mathematics is a process and result of shaping our intuitions and ideas about our internal world of activities into thoughtful models which enable us to understand and control better the whole reality. By "internal world of activities" I mean the world that would disappear if we became extinct as a species and that consists of activities over which we have strong control and which we organize and design by our human measure (e.g., movements in space, grouping and arranging small objects, writing on paper, talking, painting, playing music, etc.). It is from these concrete activities that the idea of an idealized mathematical world (model, theory) emerges, the world that expands and supplements the internal world of activities. Mathematical truths are not truths about the external world but specifications (formulations) of a mathematical world. Unlike scientific theories that are true or false about something, mathematical theories are good or bad for something.

For simplicity, I will explain this process of creating a mathematical model on the paradigmatic example of natural numbers. In his book (Mac Lane 1986) Sounders Mac Lane describes this process on a multitude of examples. Natural numbers are the result of modelling our intuition about the size of a collection of objects. This intuition stems from comparing smaller collections from our everyday world of internal activities. We measure a collection by process of counting, and natural numbers are objects created for counting. To start counting we must have the first number, to associate it to the first chosen object in the collection. To continue counting, after each number we must have the next new number, to associate it with the next chosen object in the collection. Conceptually, there is no reason to sort out some particular objects as natural numbers. Merely for the needs of calculation we sort out a particular realization, in the past through collections of marbles on an abacus, and today sequences of decimal numerals on paper and of bits in a computer. It means that for counting it is not important how numbers are realized, but only the structure of the set of natural numbers which enables us to count is important. It seems that they exist in the same way as chess figures, in the sense that we can always realize them in some way. However, the structure of natural numbers, as opposed to the structure of chess, brings an idealization. To be always possible to continue counting, each natural number must have the next natural number. Therefore, there are infinitely many natural numbers. So, although we can say for small natural numbers that they exist in some standard sense of that word, the existence of big natural numbers is in the best case some kind of idealized potential existence. Thus, we come to the idea of an idealized world of numbers that we cannot fully construct. We can only specify that world in a certain language. In that language we have names for numbers, predicate expressions for relations between numbers, and function expressions for operations between numbers. Language is primarily important as a carrier of abstraction. It separates what is important to us for numbers (first number, successor, predecessor, comparison, etc.) from what is not important (e.g., size of marbles if we use them for numbers, or font of decimal numbers if we use them for numbers). I would like to point out here that numbers are not abstract, but that we do abstraction with the help of language! The same is true for other mathematical objects. Furthermore, we specify the properties of this idealized world by certain claims of the language itself that we can axiomatically organize. This is necessary because, although we have the interpretation of the language, the recursively defined truth value of sentences is not a computable function due to the infinite domain of the interpretation. The axioms of natural numbers are neither true nor false, just as the axioms that would describe the game of chess would be neither true nor false. They are a means of specifying our ideas about natural numbers into a coherent mathematical model. It is the same with other mathematical models. Ultimately, they are always a combination of a partial interpretation in the world of our internal activities and additional specification by means of statements (axioms) of a language – a language by which we also achieve the necessary abstraction. The interpretation itself can be significant only up to isomorphism, as is the case with natural numbers, where only their structural properties in the counting process are important to us. But this is not always the case, and that is why the structural approach is one-sided. The best example of this is Euclidean geometry. It stems from our intuition about the space of our everyday activities. It is shown (Čulina 2018) how the idealization of these activities leads to Euclidean geometry. Thus, Euclidean geometry has a prominent interpretation in the world of our internal activities and is not determined structuralistically, up to isomorphism. Thoughtful modelling of other intuitions about our internal world of activities leads to other mathematical models. First, there is a not so big collection of primitive mathematical models ("mother structures" in Bourbaki's terminology (Bourbaki 1950)) that model the basic intuitions about our internal world of activities: intuition about near and remote (topological and metric structures), about measuring (spaces with measure), about straight and flat (linear spaces), about symmetry (groups), about order (ordered structures), etc. We use them as ingredients of more complex mathematical models. The complex mathematical models enable us to realize some simple and important mathematical ideas (for example, we use normed linear spaces to realize an idea of the velocity of change) or they have important applications (like Hilbert spaces which, among other things, describe the states of quantum systems). Furthermore, various mathematical models are interwoven. We express these connections by corresponding mathematical models too: these are secondary mathematical models that model how to build and compare structures (set theory and category theory) and in what language to describe them (mathematical logic). However, regardless of the complexity of the world of modern mathematics, its essence is an inner organization of rational cognition and rational activities in general based on the modelling of intuition about the world of our internal activities.

## 3. WHAT IS MATHEMATICS FOR THE YOUNGEST?

From this philosophical point of view on the nature of mathematics follows the answer to the question "What mathematics should we teach preschool children?". Just as the world of internal activities of adults is a source of mathematics for adults, so the world of internal activities of children is a source of children's mathematics. It manifests itself most expressively and develops best in children's play, being the key element of the play. Often the purpose of children's play is to understand the outside world ("let's play with dolls", "let's play cooking", etc.). When such a purpose is added to the play, then in the world of children, as well as in the world of adults, we have a mathematical model of a phenomenon. Children's stories themselves can be understood as mathematical models of certain phenomena. The Witch, for example, represents evil, Hansel and Gretel goodness, which, aided by wisdom, defeats evil and forgives the deceived (their father) but not the incorrigibly evil (The Witch and their stepmother). Here art and mathematics are almost indistinguishable.[6] The lesson is clear: the more play there is, the more math there is in the children's world. In addition to play, children develop mathematical skills whenever they try to organize their daily lives with the help of adults: arrange their toys and clothes, plan what they will do, etc. Thus, *children's mathematics consists of the world of children's internal activities that they eventually purposefully organize in order to understand and control the outside world and organize their overall activities in it.*[7] *We need to support a child in mathematical activities that she does spontaneously and in which she shows interest, and we need to teach her mathematics that she is interested in developing through these activities.* This answer to the question "What mathematics should we teach preschool children?" is completely in harmony with the methodological answer to the question "How do we teach children mathematics?", which is described above. Briefly, *a child's mathematics is part of a child's world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it*. I believe these answers, though general, give a clear conception of the place of mathematics in the children's world and our role in helping children develop their mathematical abilities. Having a clear conception is one of the key prerequisites to assist parents, educators, and teachers to successfully help the youngest in their mathematical development. In what follows, I will single out elements that are more mathematical in the sense that they empower children for more effective control of reality. Usually only these isolated elements are considered mathematics for children, as in the NCTM standards. In this way, the orientation and awareness

[6] Art and mathematics thus have the same source. Later they are differentiated by purpose, but this connection remains. That's why many, including me, believe that mathematics is, among other things, also a kind of art.

[7] Thereby, it is neither necessary nor possible in the child's current activities to strictly distinguish between what is and what is not mathematics.

that children's mathematics encompasses much more than these isolated elements is lost. Much more attention should be paid to the free child's play, stories, and organization of the child's daily life as part of his mathematics and the development of appropriate content. In the NCTM standards, this is not considered mathematics but an environment in which mathematical elements should be inserted. Thus, if we adhere to the NCTM standards then we limit the mathematical development of a child. The lack of recognition of these activities in math standards does not necessarily prevent the correct mathematical development of the child as these activities are naturally present in the development and upbringing of a child. However, the lack of recognition can lead to the fact that the environment, including the child herself, believes that she is not inclined to mathematics, even though she is. As for the elements that are more mathematical (in the sense described above), they of course include natural numbers to control quantities and geometry to control spatial activities. However, my conclusion, which I will explain below, is that the NCTM standards neither cover all the essential mathematical elements nor properly distribute attention to those elements they cover. My main criticism is that in preschool and primary school education numbers are too prominent and too elaborate and that other mathematical activities are unnecessarily subordinate to them, while in geometry too much importance is given to figures and bodies that reflect the world of adults more than the world of children. Reading the NCTM standards we can easily be convinced of this dominance of numbers and geometric figures. In the introductory chapter the following is written about the role of numbers in the mathematics education of children (page 32): "All the mathematics proposed for prekindergarten through grade 12 is strongly grounded in number. The principles that govern equation solving in algebra are the same as the structural properties of systems of numbers. In geometry and measurement, attributes are described with numbers. The entire area of data analysis involves making sense of numbers. Through problem solving, students can explore and solidify their understandings of number. Young children's earliest mathematical reasoning is likely to be about number situations, and their first mathematical representations will probably be of numbers." Especially for the youngest age, the following is written (page 79): "The concepts and skills related to number and operations are a major emphasis of mathematics instruction in prekindergarten through grade 2." The introductory part on geometry begins with the following text (page 41): "Through the study of geometry, students will learn about geometric shapes and structures and how to analyze their characteristics and relationships." Especially for the youngest age, the following is written (page 97): "Pre-K-2 geometry begins with describing and naming shapes." My goal is to show that with such an approach we are drastically limiting the mathematics of the children's world, hampering the natural mathematical development of a child, and risking that a child develops an aversion to mathematics. Indeed, as Tamás Varga (Servais, Varga 1971: 21) has pointed out, the real question is not at what age to teach a given area of mathematics but what to teach from *every* area of mathematics at a given age. *To answer in more detail the question "What of numbers and geometry, as with any other elements of mathematics, to teach the youngest?", we must take great care that it is not mathematics that belongs to our adult world but mathematics that fits into the children's world.* A detailed and complete answer to this question is, of course, beyond the scope of this article and beyond my capabilities. Finally, it is an answer that necessarily changes over time. Below I will highlight some elements that I consider to be particularly important in the mathematical development of preschool children and make some remarks on the NCTM standards.

## 4. PRIMARY MATHEMATICAL ELEMENTS

### 4.1. SETS, RELATIONS, AND FUNCTIONS

The building blocks of modern mathematics are sets, relations, and functions. They are used to build, connect, and compare mathematical structures. That is why the creators of the "new mathematics" believed that these elements must be at the very basis of mathematics education. So, they thought that the teaching of the youngest should start with these elements – which proved unsuccessful. The reason is simple to me: these concepts are foreign to the children's world. The concept of set derives from the grouping and classification of objects. However, while it is natural for children to work with concrete objects, it is not natural for them to work with abstract sets of objects. For example, a child will naturally group blue objects. She will be able to tell which object is blue, but she will have a problem if we ask her what it means "to be blue". In other words, she knows how to use the predicate "to be blue" but she cannot say what it means "to be blue". It is the same with other predicates. A child learns to use them correctly in classifying objects, but they themselves are not the object of her activities. We could go further: a child learns to use language, and with the help of language to articulate and structure her activities, but language itself is not the object of her activities at that age. Reflection on language and thinking comes mostly later. Since sets are determined by one-place predicates, relations by multi-place predicates and functions by function expressions, using language the child uses sets, relations, and functions in working with objects, but they are not the objects of her activities. Nina will talk about objects on the table and not a set of objects on the table. She will say that Ezra and Nina are cousins, but she will certainly not say that they are in a relationship of "being a cousin". She will say that Anja is Ezra's mother but not that Anja is a value of the "mom of" function applied to Ezra. *Instead of telling them about sets, relations, and functions, we need to teach children to perceive and construct concrete sets, relations, and functions.* This is what of these concepts, in my opinion, should be taught at this age. And the children's world is full of concrete examples of sets, relations, and functions. Children learn sets by grouping and classifying objects, relations by comparing objects, and functions through concrete actions over objects. All these activities are included in the NCTM standards. But that is not enough. Such important concepts require much more attention and the development of the wider range of educational activities. The "new mathematics" movement has given us a wealth of material from the field that we can, taught by history, easily transform into modern standards. For example, why stop at a comparison relation that is usually associated with some future acquiring of measurements (smaller – bigger, lighter – heavier, etc.) or an equivalence relation (same height, same shape, same color, etc.)? Why not use graphs to represent other relations? Graphs allow children to visually analyze the entire menagerie of relations from their world. Such a presentation of relations is very striking. Willy Servais (Servais, Varga 1971: 97) writes: "Arrow graphs are used to represent binary relations by sets of arrows […] The finished graph, being formed of arrows, preserves the memory of the dynamic operation involved in drawing it. […] They are really perceptual drawings fulfilling an abstract purpose. Colored graphs have made a powerful contribution to the elementary understanding of relational notions […]." E.g., we can paste or draw the characters on paper and connect them with arrows: blue for "to be a mom of", red for "to be a dad of". In this graph, children can explore family relationships; for example, find all a person's grandparents, or all her siblings, etc. Thereby, I think it's important to represent people on graphs by pictures and not by names. In my opinion, *writing and reading should not be present in mathematical content at this level because children are not fluent in these: writing and reading add unnecessary burdens and bring additional abstraction that destroys the simplicity of basic mathematical content.* We must not take written content lightly into mathematical activities. The NCTM standards do not take care of that. Furthermore, just as we can expand the mathematical content associated with relations, we can also expand the mathematical content associated with sets and functions. E.g., we can introduce operations with sets, not directly but by merging language conditions using connectives "not", "and" and "or". Thus, we teach children the correct logic of language, as demonstrated by Zoltán Pál Dienes in a lesson in logic (Servais, Varga 1971: 38–46). The NCTM standards describe various activities with functions (matching, patterns, geometric transformations, symmetries, etc.), but why not add functions that are constantly present in the children's world, such as "mom of" and "dad of", which can be combined in interesting ways for children, for example, using the graphs described above? Or movements in space (forward, backward, left, right, etc.) which can also be combined in interesting ways, for example, to discover which composition of movements can undo two steps forward, turn right and three steps backward, or to discover different compositions of movements that lead to the same result (the final position and orientation of the body).

4.2. GEOMETRY

Next to these basic mathematical elements are the mathematical elements that arise from the child's movement, navigation, and construction in space. This includes distinguishing directions and rotations, along with the "amount" of movement in a direction or in rotation. With their development, the child establishes control in space. These activities are described in the NCTM standards, but I think they are far more important than learning geometric shapes which the NCTM standards give priority to. Of course, the figures are present in the surrounding area. But it is a space designed by adults. When we transfer these figures into children's space, we must be aware that these figures do not have the same importance in the children's world as in the adult world. My limited experience has shown that in the children's world, circles, triangles, rectangles, etc., are not as prominent as they are represented in the NCTM standards. For example, Nina uses them only in the construction of patterns that are interesting to her, or they are attractive to her because of their possible symmetry. But she doesn't really care how many sides a figure has, which figure has more sides, etc. She only learned to recognize a rhombus, just because that word was interesting to her. But she showed no interest in identifying which properties characterized the rhombus in relation to other figures. I can't imagine a motivation in the children's world that would lead to identifying and analyzing the properties of geometric figures. My thesis is that children simply use figures at the preschool level but do not analyze them, just as they use the predicate "to be blue" and do not analyze it. *Children's space is primarily a space of their movements, navigation in space, and constructions in space, and the development of these abilities should be emphasized in their geometric upbringing.* In developing these abilities today, physical education helps them far more than mathematics education standards.

What is still important about geometry at this level, and which in my opinion is not adequately represented in the NCTM standards, is that geometry provides great opportunities for visual representation of problems by which a child can create mathematical models of various situations. Ordinary drawing of an elephant, for example, is the creation of a mathematical model of an elephant. Here one can follow how the child creates an ever-better model of an elephant over time, even varying the model depending on what interests her in the elephant. We can draw a strong analogy of these children's models with the mathematical models used by adults. These children's models are the first steps in modeling increasingly complex situations. Not to mention that in this way children develop a sense of space and control of lines and shapes in space, especially if they model not on paper but with some material in space. A step forward is sketching the space in which a child lives, from a sketch of the room to a map of the entire area in which she moves, as well as sketching her movement in that space using straight or curved arrows. *Making spatial maps as well as using ready-made maps and solving various problems with* *the help of maps is very important for the development of the child's mathematical abilities* and should be given more importance and more attention in mathematics education. This is very well recognized in *The National Geographic Network of Alliances for Geographic Education* (National Geographic 2022).

### 4.3. NUMBERS

Numbers are the oldest and still the most important mathematics. However, in my opinion, *natural numbers are too much imposed on the children's world and as such overshadow other mathematical content – they can even turn children away from mathematics due to their more pronounced formal aspect*. That is why numbers should be treated more carefully with the youngest than is the case now. My suggestion for preschoolers is as follows. By comparing sets by establishing a 1–1 connection between their objects, children turn their intuition of quantities into a precise mathematical model of comparing sets (it is better not to mention sets) – where there are more, where there are fewer, and where there are equal objects. The next step is to introduce numbers and a counting process that establishes a 1–1 connection with the initial segment of the set of numbers, and thus the quantities are represented by numbers. *At this level, natural numbers for children are nothing but spoken words that have a certain order in speaking*. In the Croatian language we have a series of words: "jedan, dva, tri, …". When children in Croatia learn English, they easily replace Croatian numbers with isomorphic English numbers: a new set of spoken words: "one, two, three, …". It is important to emphasize that children's numbers are always concrete objects, spoken words, and not, for example, "equivalence classes of sets according to the relation of equipotency" as the creators of "new mathematics" tried to present them to children. Today it is often imposed on children that numbers are (represented by) written signs (numerals). In my opinion, such an approach is wrong for several reasons. First of all, numerals do not have the natural order that spoken words have in chronological order, which is crucial for the counting process.[8] Furthermore, they are symbols and as such introduce at this level unnecessary abstraction into the counting process. In addition, they require a certain child's reading and writing skills, which, as I pointed out above, is a complex process that unnecessarily burdens the mathematical content. By counting, children can easily compare sets of objects by comparing the associated numbers: which numbers occur first and which later in the number sequence. Addition and subtraction of small numbers at this level can be done by adding and subtracting sets of objects that they represent, but not directly by operating with numbers. Direct operations with numbers (apart from the operation of taking the next number)

[8] This is in line with Kant's well-known claim in the *Prolegomena* that arithmetic "forms its concepts of numbers through successive addition of units in time".

not only require that children know how to write and read numbers, but they are of a formal nature which in my opinion is not part of the children's world at that age.

### 4.4. A NOTE ON OTHER MATH ELEMENTS

There is a whole series of other mathematical elements that, in my opinion, need more attention than currently given in the standards, which I will not deal with in this article. These are, for example, simpler mathematical structures (they can be developed through games that do not have to be competitive games but also cooperative games), graphs (to represent spatial networks, relations, states and changes, etc.), recursion (basic elements plus construction rules), topology (dressing, knots, transformations in clay, stretching rubber, etc.), chance (games with an element of chance), change (dynamics of movement and activities), etc.

## 5. BACKGROUND MATHEMATICAL ELEMENTS

In addition to primary mathematical elements, attention should be paid to secondary mathematical elements, elements that are present in all mathematical activities. Some of the elements have already been mentioned above: these are sets, relations and functions that appear in the children's world as primary mathematical elements through concrete examples. Then, there are abstraction, representation, procedural activities (algorithms), logic and language. But in my opinion, language is the most important, so I will dwell on it, especially since it includes both abstraction and logic. Representation has already been mentioned in the context of geometric representation of problems.

### 5.1. LANGUAGE

*Language elements are concrete means from our world of internal activities by which we control reality*. *Thus, language means form a very powerful mathematics*. By choosing words in a situation, we do an abstraction, extracting from that situation what interests us and abstracting the rest. It is an essential mechanism that helps us deal with the complexity of the world. Furthermore, we use words to control and structure the aspect of the situation that interests us. Through noun expressions we control objects, through predicate expressions we control, and I would say we refine and create concepts[9]. Thus, *language itself is an important type of mathematics that should be developed at the preschool age as well*. Like us, a child

[9] In Čulina (2021) the key role of language in our rational cognition and thinking is described.

manages to control and understand reality through language. That is why we help her a lot in mathematical development whenever we read her stories, when we listen to her talk, and when we encourage her communication with other children and adults. Of course, this attention to language development should also be nurtured in the child's mathematical activities. I would like to mention once again that at that age, *language is the means of the child's activities and not the subject of his activities.* By helping a child to develop language in a given mathematical activity, we help her to learn abstraction and to clarify the concepts or meanings of words – to clarify her mathematical means. E.g., by pointing her to triangles and quadrilaterals in composing tangrams we help her to abstract irrelevant elements (color, type of material, …) and single out relevant elements (shape and dimension) to solve problems. We also help her to specify the concept of triangle, that at some point both equilateral and right triangles are triangles, and that a parallelogram is not a triangle. In short, *by refining the language, the child refines his mathematics*. Furthermore, *by using language, the child opens the way to the idealized mathematical world that arises from her activities, thus expanding her mathematics*. This step is not a problem for the child either. Just as she uses language to specify the story of Snow White and the Seven Dwarfs, she uses language to specify the world of "all numbers". The NCTM standards do not recognize language as a very powerful mathematics and as a means of building idealized mathematical worlds, but they do recognize the importance of language as a means of clarifying and communicating mathematical activities.

### 5.2. LOGIC

No matter how we look at logic, it always manifests as the logic of a language. Thus, *by acquiring a language, children also acquire logic*. I have already mentioned the use of connectives in classifying objects using complex conditions. My experience with Nina showed me that children learn the meaning of negation ("I'm not going to kindergarten!") and of conditionals (Me: "How can I help you stop your knee hurting?", Nina: "If I watch cartoons, it will stop my aching knee.") very quickly, and somewhat slower the meaning of conjunction and disjunction. Children also understand the meaning of quantifiers ("Macarena is always angry", "Is anyone here?"). Logical inference is not foreign to them, especially when it works in their favor (Grandma: "Santa Claus only brings gifts to good children", Nina: "Then Ezra will not get a gift", Grandma: "Why?", Nina: "Because he was not good: he hit me." – there are connectives and quantifiers in this conclusion). Furthermore, if there is inconsistency in the story, a child immediately registers it.

And consistency is the equivalent of logical reasoning[10] (Me: "What's your doll's name?", Nina: "Aurora", Me: "Wasn't her name Julia yesterday?", Nina: "Yes, but she's constantly changing her name."). Although the NCTM standards emphasize reasoning as a separate process in mathematical activities, the standards limit it to the process of establishing mathematical claims, and even in such a limited context, the view of children's reasoning is very limited. What is written in the NCTM standards on page 122 – "Two important elements of reasoning for students in the early grades are pattern-recognition and classification skill" – may be appropriate for chickens but certainly not for children who are full of imagination. The NCTM standards do not recognize children's thinking as separate mathematics that develops through all children's activities, especially through stories and fantasies, and not only in mathematical activities, nor do they recognize the overall richness of children's thinking. On the contrary, it is very important to encourage children to retell or invent stories and events themselves, to discuss stories and events with each other or with us, to look for reasons for certain behaviors or events, and to draw consequences from available information.

### 5.3. PROCEDURAL THINKING

*Procedural thinking* (*how to achieve something*) *is more appropriate to the dynamics of the children's world than declarative thinking* (*what is and what is not*). However, these procedures should be meaningful and expressed in spoken and pictorial language. The refinement of procedures should be gradual with the awareness that in this way freedom is lost but efficiency is gained. Finally, adults don't really like detailed instructions, but only general instructions that leave us a lot of space for our own creation. In my limited experience, this is even more present in children. *The transition to formal procedures, such as algorithms with numbers, is a demanding transition, because formal procedures involve writing, and they lose content, so they should not be rushed*. The NCTM standards deal only with formal procedures with numbers. As formal procedures are not appropriate for preschoolers, the procedural thinking of the youngest is not present at all in the NCTM standards. This omits one important mathematical component of the child development. It can be developed very efficiently through nursery rhymes, songs, spatial movement instructions, cooking recipes, etc. For example, with the help of "The Enormous Turnip" folktale, children learn the concept of iteration in problem solving (programming loops) and with the help of "Pošla koka na Pazar" (English translation: "When Hen Was on Her Way to the Fair"[11]) South Slavic

---

[10] In first-order logic, from a given set of assumptions a conclusion logically follows if and only if the set of assumptions together with the negation of the conclusion is inconsistent.

[11] I only know of the English translation in the book (Stanić 2018).

folktale, children learn the concept of reductive problem solving (subroutine calls in programming). The development of the procedural component in children is also important due to the increasing importance of software in modern society. If we leave out technology, programming is, from a conceptual point of view, part of mathematics. Praiseworthy is the emergence of simple programming languages and environments, such as Scratch (Scratch Foundation 2022), in which children can easily and vividly create characters, program their behavior, and compose stories. All this is an important part of mathematics for the youngest to which adequate attention should be paid.

### 5.4. PROBLEM SOLVING

And at the end, an essential component of mathematics is that it has a purpose: to be a tool of our rational cognition and rational activities in general. This is true for both adults and children. Only *the purpose of children's mathematical activities must be incorporated into their world.* Just as all human civilization has developed mathematics as a means of solving various big and small problems, and just as individuals are developing it, in the same way *children in their children's world need to build their mathematics by solving problems from their world. As in the world of adults, this purpose in the world of children gives mathematical activities integrity – a natural framework for their development.* This component, which is usually called "problem solving", must be kept in mind when helping a child to develop mathematical skills. This can be solving problems arising from the organization of the child's daily activities (placing goods in drawers), arising from play (how to assemble a crane from Lego bricks) or integrated into the world of a story (e.g., the story of the wolf, goat, and cabbage). Counting on its own can be fun, but it only gets real meaning when counting controls whether all the bears are present at the morning review of stuffed animals.

REFERENCES

Bilbao (2020): A. Bilbao, *Child's Brain Explained to Parents*, Plataforma.

Bourbaki (1950): N. Bourbaki, The architecture of mathematics, *The American Mathematical Monthly*, 57(4), 221–232.

Carrier et al. (1962): G. F. Carrier, R. Courant, P. Rosenbloom, C. N. Yang, H. J. Greenberg, Applied mathematics: What is needed in research and education: A symposium, *SIAM Review*, 4(4), 297–320.

Cantor (1883) G. Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, Teubner.

Čulina (2018): B. Čulina, An elementary system of axioms for Euclidean geometry based on symmetry principles, *Axiomathes*, 28(2), 155–180.

Čulina (2020): B. Čulina, *Mathematics – an imagined tool for rational cognition*, https://philpapers.org/archive/CULMA.pdf, unpublished.

Čulina (2021): B. Čulina, The language essence of rational cognition with some philosophical consequences, *Tesis (Lima)*, 14(19), 631–656.

Dedekind (1888): R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig: Vieweg.

English, Mulligan (2013): L. D. English, J. T. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning*, Dordrecht: Springer Science+Business Media.

Ernest (1997): P. Ernest, *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press.

Hersh (1997): R. Hersh, *What is Mathematics, Really?*, Oxford University Press.

Lascarides, Hinitz (2000): V. C. Lascarides, B. F. Hinitz, *History of Early Childhood Education*, Falmer Press.

Mac Lane (1986): S. Mac Lane, *Mathematics, Form and Function*, New York, Berlin, Heidelberg, and Tokyo: Springer-Verlag.

Moss et al. (2016): J. Moss, C. D. Bruce, J. Bobis, Young children's access to powerful mathematics ideas: A review of current challenges and new developments in the early years, In: L. D. English, D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education*, third ed., 153–190, Abingdon, New York: Routledge.

National Geographic (2022): National Geographic, *Spatial thinking about maps*, https://www.nationalgeographic.org/media/spatial-thinking-about-maps/, Accessed 2022/4/24.

NCTM (2000): NCTM, *Principles and Standards for School Mathematics*, The National Council of Teachers of Mathematics.

Phillips (2015): C. J. Phillips, *The New Math: A Political History*, Chicago: The University of Chicago Press.

Scratch Foundation (2022): Scratch Foundation, *Scratch*, https://scratch.mit.edu/, Accessed 2022/4/24.

Servais, Varga (1971): W. Servais, T. Varga (Eds.), *Teaching School Mathematics*, Middelsex, UK: Penguin Books – Unesco.

Smith (2001): S. S. Smith, *Early Childhood Mathematics* (second e.d), Needham Heights, MA: Allyn & Bacon.

Stanić Rašin (2018): I. Stanić Rašin, *When Hen Was on Her Way to Market: A Folktale-Inspired Story of Manners and Illustrated Nursery Rhyme*, Arlington, MA: Perlina Press.

Stone (1961): M. Stone, The revolution in mathematics, *The American Mathematical Monthly*, 68(8), 715–734.

Boris Čulina

Veleučilište Velika Gorica
Departman za matematiku

## ŠTA JE MATEMATIKA ZA NAJMLAĐE?
## (Šta je stari matematičar naučio o matematici od svoje unuke Nine)

*Rezime*: Prema filozofiji matematike opisanoj u radu, matematika je proces i rezultat oblikovanja intuicije i ideja o našem internom svetu aktivnosti u misaone modele koji nam omogućuju da bolje razumemo i kontroliramo celi svet. Pod „internim svetom aktivnosti" podrazumevam svet koji se sastoji od naših aktivnosti nad kojima imamo izrazitu kontrolu i koje organiziramo po vlastitoj meri (npr. pokreti u sigurnom prostoru, grupiranje i raspoređivanje malih objekata, prostorne konstrukcije i dekonstrukcije s malim objektima, govor, pisanje i crtanje po papiru, oblikovanje i transformisanje manipulativnog materijala, slikanje, pevanje i sl.). Iz tih konkretnih aktivnosti nastaju idealizirani matematički svetovi (modeli, teorije) koji proširuju i nadopunjuju interni svet aktivnosti.

Iz takvog gledanja na matematiku proizlazi i odgovor na pitanje „Koju matematiku treba da uče predškolska deca?”. Kao što su interne aktivnosti odraslih izvor matematike odraslih, tako su i interne aktivnosti dece izvor dečje matematike. One se najizrazitije ispoljavaju i najbolje razvijaju u dečjoj igri – štoviše, one su sama osnova dečje igre. Često je svrha dečje igre razumevanje vanjskog sveta (npr. „Igrajmo se doktora”). Kad se takva svrha doda igri, imamo u dečjem svetu matematički model istraživanog fenomena. Pouka je jasna: što je više igre, to je više matematike u dečjem svetu. Pored igre, deca razvijaju matematičke sposobnosti kadgod pokušavaju organizirati svakodnevni život uz pomoć odraslih (npr. rasporediti svoju robu po ladicama). Tako se dečja matematika sastoji od sveta dečjih aktivnosti koju oni eventualno svrhovito organiziraju da bi razumeli i kontrolirali vanjski svet i organizirali svoje delovanje u njemu. Ovakvo gledanje je posve u skladu s ustaljenom metodologijom obrazovanja po kojoj matematičke aktivnosti deteta moraju biti deo njegovog sveta: imati motivaciju, značenje i vrednost u dečjem svetu, a ne izvana, u svetu odraslih. Ukratko, dečja matematika je deo dečjeg sveta a ne van njega, i detetu pomažemo da razvija matematičke sposobnosti u kontekstu njegovog sveta a ne van njega.

U odnosu na ovakvo gledanje na dečju matematiku, uspostavljeni standardi matematičkog obrazovanja dece su preuski: niti pokrivaju sve značajne matematičke aktivnosti niti ispravno raspoređuju pažnju među aktivnostima koje pokrivaju. Previše se pažnje posvećuje brojevima, svi drugi matematički sadržaji se podređuju brojevima, dok je u geometriji previše pažnje dato geometrijskim likovima i telima, koji više pripadaju svetu odraslih nego dečjem svetu. U članku su opisani matematički elementi koje bi bilo poželjno da deca razvijaju, a kojima u standardima nije dana dovoljna pažnja ili nisu ispravno obrađeni. To su a) skupovi, relacije i funkcije, b) kretanje, navigacija i konstrukcije u prostoru, c) vizuelna reprezentacija problema, pogotovo pravljenje prostornih mapa, d) jezik, e) logika i f) proceduralno razmišljanje.

*Ključne reči*: matematika za predškolce, standardi matematike za predškolce, NCTM standardi, pokret „nove matematike”.