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Proceedings of the Training Conference

History of Mathematics in Mathematics Education

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THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

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CONTENT

PREFACE

Scientific research in the field of mathematics education experiences a lack of institutional and financial support in some Western Balkan countries. Mathematics experts and researchers face different obstacles in their attempt to establish regional research communities in the area of mathematics education and to integrate themselves into the existing international research community.

The training conference titled *History of Mathematics in Mathematics Education* was held at the Faculty of Education, University of Kragujevac, from 26th to 30th October 2018. The main goal of the conference was to investigate the possibilities of integrating the history of mathematics into primary, secondary and higher educational settings and exploring historical traditions in mathematics education. The broader, overaching aim was to enable scholars from neighbouring Western Balkan countries to engage in common research and establish or re-establish relationships between their mathematics education communities.

The Conference was organized by the Faculty of Education in Jagodina, University of Kragujevac and Teacher Education Faculty, University of Belgrade, with the support of the European Society for Research in Mathematics Education (ERME). There were in total 28 participants from the following seven countries: Macedonia (2), Bosnia and Herzegovina (3), Croatia (1), Montenegro (1), Serbia (19), United Kingdom (1) and the USA (1). The Conference was chaired by an external expert, Dr Snezana Lawrence, Middlesex University, London, United Kingdom.

These electronic Proceedings are one of the results of the Training Conference. They comprise eleven articles which focus on various themes related to the history of mathematics and its application in the mathematics classroom. The authors give a rationale for their studies, introduce a theoretical framework and propose a brief description of research methodology. The full research papers are planned to be published in a special issue of *Teaching Innovations*, a peer-reviewed journal of Teacher Education Faculty, University of Belgrade, Serbia, in 2020.

In the end, we would like to express our gratitude to the European Society for Research in Mathematics Education for recognizing the importance of organizing this kind of an event and for supporting it. We also express our gratitude to all the participants of the Conference, particularly to those who submitted their papers to be published in these Proceedings.

Editors

MATHEMATICS, ITS HISTORY AND TEACHING

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The history of mathematics, as a part of general human culture, has a wide subject of research. It represents a base of modern scientific methodology and is one of the most important sources of a thinking process. It researches the sources and ways of generating mathematical concepts and theories, points to the extent these concepts and theories are connected with practice, describes the development of aspirations for generalizing and proving scientific assumptions, etc. The facts that are studied in teaching mathematics originated many centuries ago, and can, therefore, be understood only in a historical context and period. The paper will consider the concept of mathematics, point to the periodization of its development and connect the history of mathematics with teaching.

Keywords: History of *mathematics, educational role of mathematics, wider introduction of the history of mathematics into teaching.*

WHAT IS MATHEMATICS

The word mathematics originates from the Greek word *mathes is* that means 'learning' or 'knowledge'. Mathematics represents a common language of all sciences, technology and other human activities. Its objects are not real. A point, a straight line, a number, a plane, etc. do not exist in the real world. The same mathematical formula can be often applied to the most varied situations in reality which have the same structural characteristics in a logical and quantitative sense. Thus, for example, linear equity $y= kx$ can reflect:

– Dependence between acceleration and force, at constant mass, i.e. $F=m \bullet a$;
– Dependence between volume and mass, at constant density, i.e. $m= p \bullet v$
– Dependence between time and path, at constant speed, i.e. $s= c \bullet t$,etc.

The issue that interests mathematics in quoted and similar examples is just a relationship between sizes of some phenomena, regardless of specific characteristics of these phenomena.

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The subject of studying mathematics has changed from a period to a period. For example, ancient mathematics of Egypt, China, Babylon and India dealt with arithmetic exclusively. Later, mathematics became a science of numbers. Presently, there are hundreds of mathematical disciplines and it is very difficult to determine its subject. The simplest way to define the subject of mathematics is to say that mathematics studies characteristics of different objects (numbers, geometrical figures, sets, equities, etc.) and relationships among them, regardless of the origin of these objects.

PERIODIZATION OF HISTORICAL DEVELOPMENT OF MATHEMATICS

Subject of the history of mathematics is an origin and development of mathematical ideas, methods and algorithms. In order to organize and present the immense knowledge of mathematics it is necessary to determine a periodization type of the development of mathematics, i.e. to determine the phases of its development. Determining periodization criteria represents a great problem. Many mathematicians determine a periodization type according to a chronological criterion. One of the most general types of periodization was determined by Kolmogorov (Kolmogorov, A.N., 1903-1987), a great Russian mathematician. A methodological principle Kolmogorov applied in his periodization type is transition from a lower degree abstraction to a higher degree abstraction. He suggests four periods of the historical development of mathematics:

1. Originating of mathematics
2. Elementary mathematics
3. Originating of mathematics with variable values
4. Modern mathematics

The first period lasted from the very beginnings of human existence till the 6thcentury B.C. In this period mathematics was of practical character exclusively; there was no proving nor generalizing or theoretical foundation.

The second period began in the 6thcentury B.C., when mathematics raised to the level of a science and when its foundations were set. Mathematics was differentiated by its subject of research and its methods. Geometry, elements of the theory of numbers, algebra and trigonometry were formed in this period. This period ended at the beginning of the18thcentury with the appearance of Descartes' coordinate geometry in 1637, when the third period began, in which new mathematical ideas originated.

In the third period there was a turbulent development of the complete human civilization. It was the time of new geographical discoveries, seafaring and astronomy developed, as well as manufacture artillery, etc. All that demanded new ideas and new mathematical solutions, elementary geometry and algebra being insufficient to fulfill new demands. Firstly, mathematics was then interested in variable values and their interdependence – functions. Analytic geometry and mathematical analysis were established.

The 19th century could be marked as the beginning of the fourth period of the Kolmogorov periodization. This period, above all, has been characterized by wideness of mathematical research subjects. New, non-Euclid's geometries were created. Number was no longer the main object in algebra in operations, but it operated with objects of a general character. Group theory and linear algebra appeared then. The sets theory influenced the general path of mathematics. New branches developed in analysis (for example, the theory of the real variable function), as well as topology, functional analysis, etc. Mathematical logics developed, and completely new fields also appeared, such as: the theory of algorithms, the theory of information, operational research, cybernetics, and discrete and final mathematics were born as well.

THE HISTORY OF MATHEMATICS AND TEACHING MATHEMATICS

- Some of educational goals which are achieved by application of mathematics in teaching are:
- Serious science is presented to children more easily through the history of mathematics. Teaching becomes more interesting and livelier.
- Pupils notice that mathematics is a product of a creative activity of the human genius during thousands of years, and not a momentary inspiration of "some clever people". Each mathematical model is a result of catching impulses of nature through centuries.
- The pupils will learn that they should endure a long and difficult path to doing creative work, and that they should make a lot of mental effort in the process, just as the previous generations used to do.
- When doing creative work, besides a series of routine mathematical actions which can be done by every mathematician, there is often at least a decisive one which is the result of a unique idea. The history of mathematics possesses a series of enlightening examples about this matter. One of the most beautiful proofs is the Euclid's proof that prime numbers are infinite.
- Biographies of great people can often help us make useful conclusions which can influence pupils in an informative and educational way.
- If mathematical theories are regarded through their final formulation only,

without their historical interpretation, pupils could get a wrong notion of mathematics as some artificial creation, which serves to wise, mental imaginations, without any connection with practice and applications in it. The importance of studying mathematics will be clearer to pupils when, through historical facts, they understand that mathematics, since its origin, has had one of the most important roles in all activities of a human being.

- A class of mathematics along with treating historical facts receives a cultural dimension.
- Pupils can easier acquire knowledge if they follow a historical way of developing a concept. The history of mathematics should depict to the pupils the origin of a concept at its source and point to the natural and logical way to its highest abstraction. It would be very useful for the pupils to understand the way the creators used to go along, how the process from the idea to announcing the results has gone.

The teachers can educate themselves through the following resources on the history of mathematics in teaching:

1. *Seminars*: it is suggested that professional associations should organize seminars in which teacher practitioners would be educated in connection with usage of the history of mathematics in teaching.
2. *Magazines*: it is suggested that the themes from the history of mathematics should be included much more in the magazines for pupils. These themes should be in line with the curriculum. The history of mathematics themes should also find more place in magazines intended for the teachers.
3. *The textbook*: it is proposed to write special textbooks on the history of mathematics. This kind of practice is known in Russia, France, Italy and Romania. The books should be adapted to two levels of education, i.e. elementary and secondary school. The contents of textbooks like this should completely follow the curriculum of mathematics. The teacher should also examine all historical facts related to a teaching unit that has been taught as well within the preparation for work.
4. *The lexicon* of mathematical signs and symbols can be very useful, especially at introducing new mathematical symbols. Then, from the lexicon, the historical path should be recognized, the reasons for its specific form, the person that was the first to use this symbol, the qualities of the other forms, etc.

Methodology recommendation is not to exaggerate with historical facts and not to blur the essence of the theme that has been taught. When a name of some mathematician is mentioned, it is sufficient to retell some anecdotes from his life, to enlighten his work or some concept related to his name. To help children master the concept which has been taught, a short historical development of the concept should be given to them. It is sufficient to spend up to five minutes in each class discussing historical facts; moreover, this will significantly improve the discipline during the class. Besides telling indispensable historical facts related to concepts taught in regular teaching, we can introduce elements of the history of mathematics in additional classes. The pupils should elaborate historical themes for several days, and after that, discuss the elaborated themes. Many of the same historical issues could be taught within the scope of different grades and in various contexts.

CONCLUSION

Some theoretical opinions have been briefly presented in connection with the theme of the history of mathematics and its application in teaching. A general conclusion is that the history of mathematics is presented insufficiently in teaching mathematics and that its presentation depends on the willingness of teachers to use it more widely. However, there are ways for presenting it more in teaching. The most efficient way is to incorporate elements of the history of mathematics into the textbooks of mathematics, in places where it is necessary, and to the right extent. It is necessary to study concrete examples of application of the history of mathematics in teaching mathematics and to give methodology recommendation for their application in teaching. It is also necessary to survey the teachers' attitudes to this theme and in accordance with those attitudes to make a concrete curriculum for wider introduction of the history of mathematics into teaching.

The paper also presents a short list of literature relevant to the theme, which can be used in a longer paper.

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ABU'L-WAFA PROBLEM – POSSIBLE TOOL FOR FOSTERING SUBJECT AND PEDAGOGICAL CONTENT KNOWLEDGE OF PRE-SERVICE MATHEMATICS TEACHERS

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In this paper we present the problem posed by medieval Middle Eastern mathematician Abu'l-Wafa, as well as different solutions of this problem. Solving Abu'l-Wafa's problem could be a powerful tool for building and fostering subject content knowledge and pedagogical content knowledge of pre-service mathematics teachers. In solving this problem students have to use many geometrical concepts. On the other hand, as future mathematics teachers, they have to find out when and how to engage pupils to solve this problem, considering the question of procedural and conceptual knowledge in mathematics as well as the important question of the role of proof and argumentation in mathematics classes.

Keywords: Abu'l-Wafa problem, mathematical content knowledge, pedagogical content knowledge.

INTRODUCTION

In geometry classes, apart from other problems, we solve construction problems, i.e. in given conditions, we draw a geometrical figure using only a ruler and a compass. Solving construction problems brings several benefits to pupils: they learn and use construction procedures, connect geometrical concepts, provide proofs for proposed procedures, discuss different aspects of problem's conditions and connect them with possible solutions. Furthermore, when solving this type of problems, pupils should precisely and accurately express geometrical ideas in a symbolic, graphical and textual representation.

In next chapter we will present different solutions to a problem posed by Middle Eastern mathematician Abu'l-Wafa. This problem could be posed to pupils of different ages, either in middle or high school.

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ABU'L -WAFA PROBLEM

Abû'l-Wafa Buzjani (940-998) was born in Buzjan, near Nishabur, a city in Khorasan, Iran. He learned mathematics from his uncles and later on, when he was in his twenties, he moved to Baghdad. He flourished there as a mathematician and astronomer. In his treatise titled *On Those Parts of Geometry Needed by Craftsmen,* Abu'l-Wafa described several constructions made with the aid of a ruler and a "rusty compass", a compass with a fixed angle. These included constructing a perpendicular at the endpoint of a line segment, dividing segments into equal parts, bisecting angles, constructing a square in a circle and, constructing a regular pentagon (Berggren, 2003).He was one of a long line of Islamic mathematicians who developed geometric techniques that proved useful to artisans in creating the highly symmetrical ornamentation found in architecture around the world today (Tennant, 2003).

Problem: *Construct at the endpoint A of segment AB a perpendicular to that segment, without prolonging the segment beyond A.*

The problem may be solved in different ways.

Solution 1: Abu'l-Wafa's solution (Berggren, 2003)

Procedure. On AB mark off with the compass segment AC and with the same opening, draw circles centered at A and C, which meet at D. Extend CD beyond D to E so that ED = DC. Then $\sphericalangle EAB$ is a right angle (Figure 1).

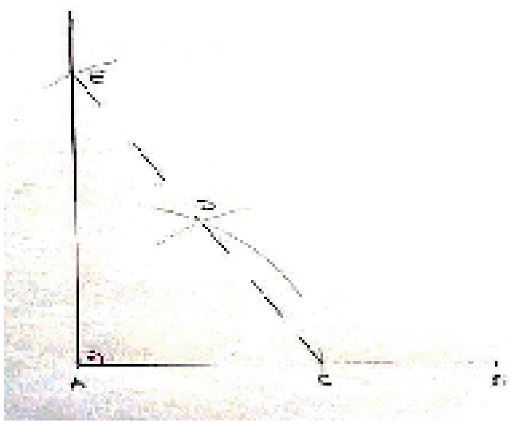

Figure 1. *Solution 1*

Proof. The circle that passes through E, A, C has D as a center since DC = DA = DE. Thus, EC is a diameter of that circle and therefore $\sphericalangle CAE$ is the peripheral angle subtended by the diameter of the circle. Each such angle is the right angle (Tales theorem about angle subtended by diameter of circle).

Remark. In order to perform this solution pupil should use notions of central and peripheral angle as well as Thales theorem about angle subtended by diameter of circle.

Solution 2:

Procedure. Using a compass construct angles: $\sphericalangle pAq = 60°$ and $\sphericalangle qAr = 60°$. Construct bisectors of $\sphericalangle qAr$. Angle $\sphericalangle pAs$ is the right angle, i.e. s in perpendicular to segment AB (Figure 2).

Proof. Angles $\sphericalangle BAq = 60°$ and $\sphericalangle qAr = 60°$ are adjacent angles. Bisectors of angle $\sphericalangle qAr$ passes through A and divide $\sphericalangle qAr$ into two equal angles $\sphericalangle qAs = 30°$ and $\sphericalangle sAr = 30°$. Angles $\sphericalangle BAq$ and $\sphericalangle q$As are adjacent angles. Their sum is angle $\sphericalangle BAs$.

$$\sphericalangle BAs = \sphericalangle BAq + \sphericalangle qAs = 60° + 30° = 90°.$$

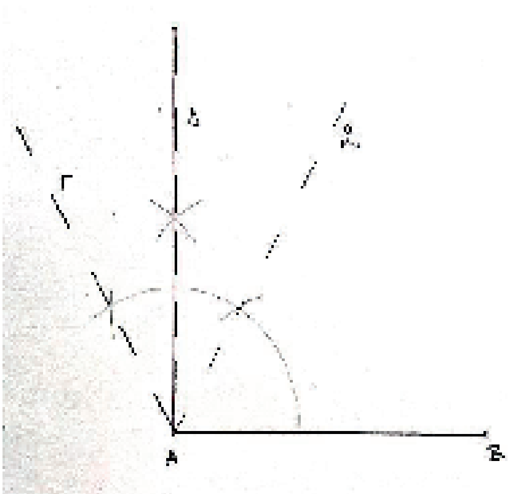

Figure 2. *Solution 2*

Remark. This solution uses construction of angles of 30°, 60° and 90° and is suitable for pupils in early grades of middle school.

Solution 3:

Procedure. Construct the bisector of the segment AB. Translate the bisector to the endpoint A of segment AB (Figure 3).

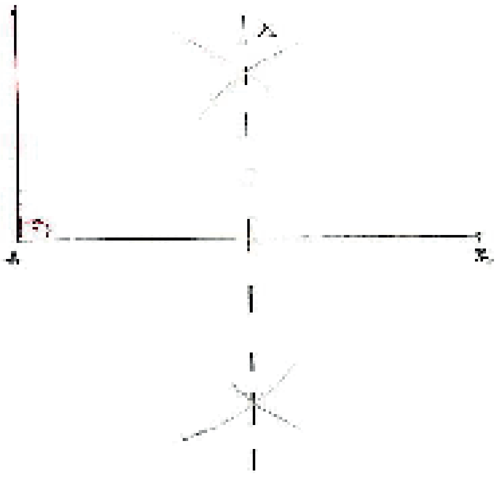

Figure 3. Solution 3

Proof. The bisector of segment AB is perpendicular to AB. Now we use a property of isometry: each translation maps a right line into a parallel right line.

Remark. This solution does not represent geometrical construction because two rulers were used to translate bisector s to the endpoint A. However, this solution is appropriate when pupils learn about isometries in the middle school.

USE OF ABU'L-WAFA PROBLEM TO FOSTER CONTENT AND PEDAGOGICAL KNOWLEDGE OF MATHEMATICS TEACHERS

Mathematics teachers have to master mathematical contents, as well as pedagogical content knowledge that will enable them to create an environment in which pupils will learn in an optimal and effective way. In other words, the teacher must be able to adapt mathematical content to the age of the student, without compromising the principle of science. In mathematics teaching and learning it is equally important to develop both procedural and conceptual knowledge among pupils. Mathematics teachers need to find a "right measure" in teaching, paying attention both to performing mathematical procedures as well as to giving explanations, argumentations and proofs. They have to promote such mathematical culture among their pupils. Using episodes from history of mathematics could help teachers to accomplish this.

Integration of the history of mathematics in teaching and learning mathematics could be justified by several arguments. Tzanakis and Arcavi (2000) point out the following five fields in which integration of the history of mathematics may be particularly relevant to support, enrich and improve the teaching and learning process:

- learning of mathematics;
- nature of mathematics and mathematical activity;
- didactical background of teachers;
- affective predisposition towards mathematics;
- appreciation of mathematics as a cultural endeavor.

CONCLUSION

Taking into account previous considerations, the Abu'l-Wafa's problem can be used to build and foster subject content knowledge and pedagogical content knowledge of pre-service mathematics teachers.

Abu'l-Wafa's problem will take a central place in our case study among pre-service mathematics teachers. The goal of this study is to foster subject content knowledge and pedagogical content knowledge of a group of pre-service mathematics teachers by using the episode from history of mathematics.

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INTEGRATING GEOMETRY AND ALGEBRA AS A WAY OF REIFICATION OF MATHEMATICAL CONCEPTS – HISTORICAL ASPECT

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The purpose of this paper is to illuminate the historical aspects of mathematics that could be used for the improvement of teaching practice. The idea that is already established in the literature is the existence of the similarities between historical development of mathematical concepts and development of concepts in the process of learning. Geometric algebra – the term which we use to describe content of the Euclid's Elements II is an important phase in the history of mathematics. On the other hand, geometry models in algebra are recognized as an important phase in the development of concepts in the learning process. The aim of this paper is to explore students' achievement in solving problems using geometry models. We also pose a question about the relationship between solving methods and students' age. If development of concepts follows their historical development, younger students should be oriented towards geometry visualizations and older to algebraic symbolism.

Keywords: Geometric algebra, visualization, history of mathematics

THE HISTORICAL DEVELOPMENT OF ALGEBRAIC CONCEPTS

Sfard (1995) uses the operational – structural model in the analyses of historical development of mathematics. She describes the development of algebra "as a hierarchy in which what is conceived operationally (i.e., as a computational process) on one level is reified into an abstract object and conceived structurally on a higher level" (Sfard, 1995, p. 16). She presents algebra learning as "a constant (but not necessarily conscious) attempt at turning computational procedures into mathematical objects, accompanied by a strenuous struggle for reification" (Sfard, 1995, p. 17). Reification is considered as a process of turning computational procedures into structural objects. This process requires the change of the

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way mathematical concept is comprehended – the processes on objects become objects themselves.

According to literature, the most frequently used phases in the development of algebra are Nesselmann's rhetoric, syncopate and symbolic algebra (according to Bagni, 2005; Kieran, 1992). However, numerous authors emphasize the Greek geometry as important phase in the historical development of algebra.

FROM EUCLID'S GEOMETRIC ALGEBRA TO THE INTEGRATION OF SYMBOLIC ALGEBRA AND GEOMETRY

The geometric algebra was based on interpretation of quantities expressed in the terms of line segments and figures. The aim of operating with these quantities is to find perimeters, areas or volumes of figures and shapes. By analyzing the historical development of algebra, Sfard highlights the procedural (operational) character of geometric algebra in the following way: "Greek geometry, with its "thinking embodied in, fused with graphic, diagrammatic representation" (Unguru, 1975, p. 76) was clearly at its structural stage, whereas algebra, preoccupied with verbally represented computational processes, could be conceived in no other way than operationally..... What algebra needed for further development was reification of its basic concepts. At this time, no better means were available to help in reification of the growingly complex computations than the palpable geometric objects." (Sfard, 1995, p. 23).

The content in Euclid's Elements was systematized in definitions, postulates, common notions and propositions. The books have geometry content, but today's interpretation of its proposition could be broader. The Second book is algebraic in nature. The Fifth book about proportions could be also understood as algebraic, with interpretation of algebraic problems in terms of geometry proportions. Books VII to IX are dedicated to theory of numbers - not the calculation procedures, but Pythagorean questions such as divisibility of whole numbers, sum of geometric progression, etc.

The procedure that could today be interpreted as algebraic, in Elements had geometric form. The expression $\sqrt{A}$ in Elements is the side of a square with area A, and expression $a \cdot b$ is area of rectangle with sides a and b. In Elements the request to determine one magnitude over others would be analogous with the construction with a compass and ruler. Kline argues that the roots of geometric algebra are related to Eudoxus (408-355 B.C.) who introduced the notion of a magnitude to denote entities like line segments, area, and volume (Kline, 1972, 48). Quantities are not assigned to these magnitudes (so Eudoxus ideas avoid irrational numbers) so this enables Antic Greeks to get generalized results. Kline's conclusion could be illustrated though the example of the fourth proposition of the Second book of Elements:

If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

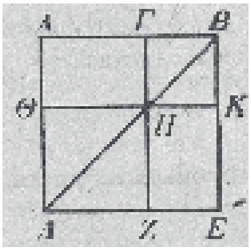

Figure 1: Euclid's Elements Book II Proposition 4.

Today, this proposition could be expressed as $(a + b)^2 = a^2 + b^2 + 2ab$ but in Elements Figure 1 illustrates its proof. The proof is in rhetorical form. We will illustrate the language in Elements through proof of one simpler proposition of Second book – Proposition 2 which states: If a straight line is cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole. Algebraic: if c = a + b then c^2 = ac + bc.

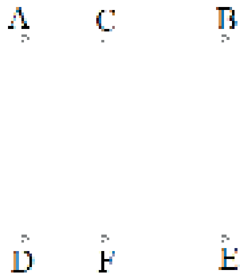

Figure 2: Euclid's Elements Book II Proposition 2.

Proof. For let the straight line AB be cut at random at the point C; I say that the rectangle contained by AB, BC together with the rectangle contained by BA, AC is equal to the square on AB. For let the square ADEB be described on AB [I. 46], and let CF be drawn through C parallel to either AD or BE. [I. 31] Then AE is equal to AF, CE. Now AE is the square on AB; AF is the rectangle contained by BA, AC, for it is contained by DA, AC, and AD is equal to AB; and CE is the rectangle AB, BC, for BE is equal to AB. Therefore, the rectangle BA, AC together with the rectangle AB, BC is equal to the square on AB. Therefore etc. Q. E. D. (Euclid, Elements, II. 2).

The development of symbolical algebra enabled mathematicians to express their ideas in a manner that is quicker and easier to communicate. Descartes (1596–1650) and Fermat (1601–1665) were the first to include symbolic algebra in geometry. They integrated geometry figures and transformations with calculation procedures. This was later called analytic geometry. Struik (1991) highlights that there are great differences between today's and Vieta's algebra. Vieta's algebra goes along with the Greek principle which treats product of length as area, and by which length only adds with length, area with area, and volume with volume. Descartes overcomes constrains that his predecessors had, which is recognized in Vieta's logistica speciosa, and considers x^3, x^2, $x \cdot y$ as line segments. Algebraic equations became relations between numbers. This was a huge step forward for mathematical abstractions.

GEOMETRIC ALGEBRA AS A SUPPORT FOR DEVELOPMENT OF ALGEBRAIC CONCEPTS

By opposing historical and ontogenetic development of algebraic thinking, Sfard (1995) emphases that difficulties that students have on different levels of learning mathematics could be close to difficulties that generations of mathematicians have experienced. She argues that there are reasons which refer to similarity of phylogenesis and ontogenesis of mathematical thinking. Katz and Barton (2007) highlight that historical motivation for developing algebra arises from the need for solving concrete, real and mathematical problems. They also argue that even if the most frequent starting points in algebra learning are generalized arithmetic and problem solving, geometry models could be used for the development of algebraic ideas. The examples are geometry concepts such as perimeter and area. Products of numbers could be represented as rectangles, and as an example, distributive property could be viewed as a statement about the rectangle from two different perspectives. These models are viewed as more concrete than real schematized object usually used in teaching practice.

It is hard to imagine that today's students are able to connect the above stated proposition (Figure 2)with distributive property $(a + b)c = ac + bc$ which is introduced in arithmetic through calculation procedures and justified in algebra through properties of algebraic structures. The rhetorical proof of the proposition gives an opportunity to students to use geometry visualizations and to construct a proof based on geometry statements. The question is if students are able to justify the proposition without the use of symbolic algebra.

Research has shown that today students and teachers tend to move to algebraic symbolism too quickly. This leads to absence of meaning of algebraic and geometric concepts and to an incomplete understanding (Kieran, 1992; Gravemeijer & Terwel, 2000). The use of mathematical formalism and absence of visualization in problem solving shows deficiencies in a methodical approach in earlier years of education.

DESCRIPTION OF THE RESEARCH

We have argued that historically, development of mathematical concepts relied on visualizations and geometry representations. In this paper we are asking if there is enough attention dedicated to visualization at the beginning of learning the basic mathematical concepts. Is the development of meaning based on geometry figures? The aim of this research is to investigate the students' achievement in solving problems by using geometry visualizations. We used propositions from the Second book of Elements as motivation for creating problems in the research. The problems are given in geometry context, but they could be solved algebraically. Historical development shows that those problems were the first solved using geometry visualizations and deduction and that the later development phase geometry transformations became proper algebraic transformations.

In this research we will analyze the responses of students of different ages. We want to answer the question if students of different ages use mathematical language which corresponds to different historical phases, i.e. if younger students tend to visually solve problems (like in ancient Greek phase) and older students tend to use algebraic transformations. The sample would be made from the students from 4^{th} and the 8^{th} grades of primary school. The test is created for the purpose of the research. The examples of tasks which will be used for the investigation of students' reification of concepts and the use of visualization is presented in Table 1.

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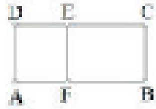
1. The side AB of rectangle ABCD is divided into two parts with length 3 cm and 5 cm. The side CD is divided also into two parts with length 2 cm and 3 cm. Calculate the area of rectangle ABCD.
2. The side AB of rectangle ABCD is divided into two parts with length *a* and *b*. The side CD is also divided into two parts with length *c* and *d*. Express the area of rectangle ABCD.
3. One side of rectangle is increased for 2 cm and the adjacent side for 3 cm. How will the area of rectangle be changed?

4. The rectangle ABCD is divided into square AFED and rectangle FBCE like in the picture. Express the area of rectangle ABCD if the area of square AFED is $a \cdot a$ and of rectangle FBCE $a \cdot b$.



5. The square with the side a is divided with two lines into two squares and two rectangles. Calculate the length of side a if one of the squares has area 25 cm^2, and one of the rectangles has area 60 cm^2.
6. The square with side a is divided with two lines into two squares and two rectangles. Express the length of side a if one of the squares has area b^2 and the other c^2. Also express the area of rectangles.

Table 1: Examples of tasks which will be used in research

Our hypothesis is that there is a tendency towards algebra from early grades. We want to investigate if students use visualization and visual problem solving, because it is an important part of the process of reification of mathematical concepts and procedures.

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HISTORY OF GRAPHICAL REPRESENTATION OF DATA

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Today it is considered that graphical presentation of data is natural and obvious. However, history of development of graphical representation of data suggests that it is a complex task. A certain level of social and scientific potential was necessary for creation and development of simple and efficient methods of graphical representation of data. In this paper we shall give brief overview of history of graphical representation of data which will be a starting point for further research.

Keywords: Charts, statistical graphs, graphical representation of data, history of mathematics

GRAPHICAL REPRESENTATION OF DATA

The beginnings of the visualisation are deep in our history. Roots of graphical *representation* can be found in cartography. According to Herodotus (books II and IV), the author of the first world map was Anaximander from Miletus (550 BC). Cartography and statistical graphics share common goals of visual representation for exploration and discovery. Cartographic visualisation is primarily concerned with representation constrained to a spatial domain, while statistical graphics applies to any domain in which graphical methods are employed in the service of statistical analysis. Beside this difference, cartographic visualisation and statistical graphics share common historical themes of intellectual, scientific, and technological development.

In the 16th century techniques an instruments for precise observation and measurement of physical quantities were well developed (Friendly, 2009). In the 17th century "the rise of analytic geometry, theories of errors of measurement, the birth of probability theory, and the beginnings of demographic statistics and political arithmetic" were observed. With statistical thinking visual thinking evolved.

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The 17th century was considered as the one in which visual thinking began. The main problems were those concerned with physical measurement of time, distance and space; for astronomy; surveying, map making, navigation and territorial expansion. In the 18th century, abstract graphs and graphs of function were introduced, along with the early beginnings of statistical theory and the systematic collection of empirical data. In 1748, the term “statistik”, before that, “statist”, “statista” and almost forty years later, in 1787, Zimmerman uses word “statistics” as it is known today (Friendly, 2009). Between 1786 and 1801, the circle diagram, the pie chart and bar chart were invented and published by William Playfair (Playfair, 1801). Playfair also improved a line graph for representation of economic data. At this time

> ... illustration in serious writing was viewed with suspicion. It would have been unthinkable to introduce pictorial material to bolster an argument where tabular presentation would have been seen as sufficient and certainly more accurate. (Spence & Wainer, 1997, p. 31)

Williams’ graphical innovations were not accepted by his contemporaries, but their importance grew after the second half of the 19th century, when Charles-Joseph Minard (1781-1870) and Jacques Bertillon (1851-1922) used some of Playfair’s methods in their cartographical work (Spence & Wainer, 2005b). The fact that graphical representation of data which he invented have the same form today, after psychological research about human perception of charts, suggests that we need to pay more attention to this visionary.

DEVELOPMENT OF NEW GRAPHIC FORMS FOR REPRESENTATION OF DATA

William Playfair (1759-1823) had wide mathematical and philosophical education. His older brother John Playfar (1748-1819), professor of mathematics (1785) and natural philosophy (1805) at Edinburgh University, taught him mathematics. According to Spence (2006), William was acquainted with work of philosophers Dugald Stewart (1753-1828) and Tomas Reid (1710-1796), whose preoccupations were perception and memory. Besides that, while working for Andrew Meikle (1719-1811) and James Watt (1736-1819), he acquired skills like drawing, using shading and labelling (Spence, 2006). All of that had a great impact on the development of the idea of necessity for graphical representation of data. He understood the advantages of graphical representation and knew that charts were a universal language applicable in science and economy. He thought that graphical representation contributed to a faster and easier reading and interpretation of data (Playfair, 1801, xii; Sachs, 2012).

William Playfair was the first to use a graphical representation for quantitative data (mainly economic) and to publish a book to contain statistical graphs. This book was his Commercial and Political Atlas (Playfair, 1986) which contained 44 charts (Funkhouser, 1937). Graphical representations before Playfair had been graphical representations (line diagrams) of functions and they had not been based on quantitative or statistical data. On the contrary, Playfair included raw data "in accompanying tabular form for many of the charts" (Spence, 2006, p. 2427), so it is clear that his line diagrams "are not mathematical function derived from theory" (Spence, 2006, p. 2427). Moreover, Playfair's graphical representations had "graduated and labeled axes; grid lines; a title; labels" (Spence, 2006, p. 2427) and other characteristics typical of today's graphical representation of data.

Playfair himself explained motivation for introducing visual displays of data in the following way:

This method has struck several persons as being fallacious because geometrical measurement has not any relation to money or to time yet here it is made to represent both. The most familiar and simple answer to this objection is that if the money received by a single man in trade were all guineas and every evening he made a single pile of all the guineas received during the day, its height would be proportioned to the receipts of that day, so that by this plain operation time, proportion and amount would be physically combined. (Playfair, 1801, according to Royston, 1956, p. 242.)

Today, "the piles of guineas" are called object graphs. It is interesting to note that, in contemporary teaching practice in elementary schools, it is common for object graphs to be used in the first stage of the development of graphic representation skills and understanding of data.

According to him, credits belong also to his brother John.

... my brother, ..., made me keep a register of a thermometer expressing the variation by lines on a divided scale. He taught me to know that whatever can be expressed in numbers may be represented by lines. The chart of the thermometer was on the same principle whit those given here (in Atlas), the application only is different. (Playfair, 1805, xvi; according to Royston, 1956, p. 242 and Sachs, 2012, p.4)

Playfair was the first to use charts to "support his economic and political arguments" (Spence, Fenn, & Klein, 2017, p. 20-21; Spence, 2005a). It seems that he was aware that his graphical forms would not be easily accepted and understood, so he gave detailed descriptions of his charts.

Another Playfairs contemporary saw the value of graphical representation of data. A. F. W. Crome(1753-1833) was a professor of Statistics and Public Finance at Giessen University from 1786 until 1831. Crome first applied geometric representation as an aid to teaching (Royston, 1956). In 1820 Crome for the first time used circle graphs for representation of data. Playfar's circle diagram is clear and simple; on the contrary, Crome's one is similar, but complicated. Playfair's circle diagram had been published before Crome's, but there is no evidence that Crome had used Playfair's idea. According to Royston (1956) it is possible that Playfair and Crome worked indepedently.

GRAPHICAL REPRESENTATION OF DATA SINCE THE 19TH CENTURY UNTIL TODAY

In the 19th century, statistical graphics experienced a big revelation. At the beginning of this period (in 1801), the first large-scale geological map of England and Wales was created, setting the pattern for geological cartography, and founding stratigraphic geology.

By the second half of the 19th century all necessary tools for data representation were discovered, statistical offices were established throughout Europe and some statistical theories from the past started to collect and represent large sets of data. This period is characterized by a several breakthrough in the representation of data such as:

Representation of data started to be used more often in different everyday situations and scientific fields: mining meteorology, politics, demography, advocacy, medicine, transport, traffic, etc. New types of graphs were used for the first time or in a new way: Coxcombs, Semi logarithmic grid, stereogram, Semi-graphic table to display a data, table by shading levels, Bilateral histogram, two-variable colour map, divided square in the modern form for data representation, Polar diagrams and Star plots, Pictogram, Alignment diagrams, Anamorphic maps, Area rectangles on a map, etc. The term graph was introduced, referring to diagrams in chemistry; the first mature attempt at a systematic classification of graphical forms is made by Statistical Society of London, which gave a comprehensive review of all available statistical graphics and several international conferences and exhibitions were organized. The first international statistics conference was organized in 1853 by International Statistical Institute Belgium. Statistical diagrams started to be used in school textbooks. Diagrams were incorporated in museums (Outlook Tower, Edinburgh, Scotland, 1892).Consequently, the second half of the 19th century is often called Golden Age of data graphic.

At the beginning of the 20th century, enthusiasm for visualization was reduced due to the rise of statistical models in the social sciences. Only few new diagrams were presented (Butterfly diagram, Hertzsprung-Russell diagram, Gantt chart, Path diagram, Nomogram, Ideograph). By the mid-1930s previous graphical methods found new application and use in government, commerce, astronomy, physics, biology and other sciences and entered textbooks (the USA and England). For the first time several scientific societies published standards for graphical representation, and commercial correspondence courses in graphical methods started.

After WWII data visualization experienced its own rebirth in mid 1960s thanks to the three significant developments: publication of papers "The Future of Data Analysis" and "Semiologie Graphique", and the beginning of computer data processing. Invention of different software tools, languages (C, UNIX, etc.), display and input technologies, provide new ways for expressing and implementing data graphics, which is in progress even today. Nowadays it is very hard to provide an overview of the most recent developments because they are spread across a wider range of disciplines.

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THE RELATIONSHIP BETWEEN IMAGES AND CONCEPTS IN THE INITIAL GEOMETRY TEACHING

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Drawing upon Fischbein's theory of figural concepts, the starting point of the paper is the use and value of the history of geometry in mathematics education. Fischbein's theory is mainly based on a hypothesis that geometry deals with mental entities, the so-called geometrical figures, which simultaneously possess conceptual and figural properties. Fischbein called the geometrical figures figural concepts because of their nature. We shall analyze the internal tensions which, according to Fischbein, may appear in figural concepts because of their double nature, developmental aspects and didactical implications. In doing so, we have created a room for further research, the goal of which will be to examine the pre-service primary school teachers' geometric reasoning regarding the correlation between figural and conceptual properties of geometric objects in order to obtain a framework for creating didactic situations in which a figure and a concept would merge into a unique mental object.

Keywords: History instruction, geometric object, geometric reasoning, notion of figural concept, pre-service primary school teachers.

THEORETICAL BACKGROUND

Drawing upon the theoretical framework of development of students' geometric thinking (Dreyfus, 2014; Đokić & Zeljić, 2017), students' geometric reasoning (Đokić & Zeljić, 2017) and Fischbein's theory of figural concepts (Fischbein, 1993), the authors of the paper explore the use and value of the history of geometry in mathematics education (Đokić, 2017; Gulikers & Blom, 2001) as the starting point in their research. The main hypothesis of Fischbein's theory (1993) is that geometry deals with mental entities, the so-called geometrical figures that simultaneously possess conceptual and figural properties. Fischbein called the geometric figures *figural concepts* because of their nature. We shall analyze the internal tensions according to Fischbein which may appear in figural

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concepts because of their double nature, developmental aspects and didactical implications.

DEFINITION AND IMAGE

The relationship between a definition and an image is viewed differently in empirical sciences and mathematics (Fischbein, 1993). In empirical sciences, the definition is ultimately determined by the properties of the objects defined, whereas in mathematics the definition, created directly or by deduction, is the property of the corresponding objects. Therefore, the interpretation of the figural components of geometrical figures should entirely be subjected to formal constraints. This idea is not always clearly understood and students often tend to overlook it. The difficulty with manipulating figural objects lies in a tendency to neglect the definition due to figural constraints which are the main obstacle to students' geometrical reasoning.

The First Aspect of Clarity: Figural Concepts – Conflict Situations

Fischbein offers an example how to help students cope with this type of conflict situation. Students may not be able to draw correctly the altitude of the triangle ABC from vertex B (Figure 1), and they will draw BD instead, despite the fact that they know the definition of the altitude of the triangle.

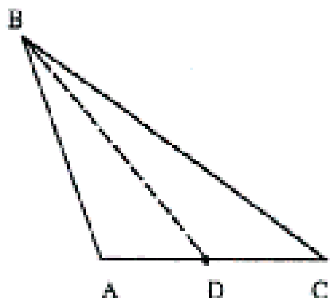

Figure 1: The problem with drawing triangle altitude according to Fischbein

The students should know the definition and solve the task by following it, and not according to what they think they see in the image. Many similar examples should be systematically used in the classroom, so that the domination of the definition over the image in interpretation of figural concepts becomes absolutely clear to all students. Fischbein provides another example. In this task, students have to compare the set of points in the segments AB and CD, and to resolve the conflict between two claims: that CD has more points than AB and the claim that two sets of points AB and CD are mutually equivalent (Figure 2).

Figure 2: Comparing the sets of points according to Fischbein

The integration of conceptual and figural properties in a unique mental structure, with the predominance of conceptual constraints over the figural ones, according to Fischbein is not a natural process. Fischbein claims that "the processes of building figural concepts in the students' minds should not be considered a spontaneous effect of usual geometry courses" (1993, p. 154–156). This process requires teachers' continual, systematic, and committed involvement. Conflict situations should serve to teach students how to follow carefully the requirements set in definitions, which sometimes can run contrary to what the images seem to represent (or impose).

The Second Aspect of Clarity: Figural Concepts – the Concept of Locus

Fischbein asks a question whether a circle is only an image (i.e. a spatial representation). According to Fischbein, its existence and properties are imposed by an abstract, formal definition. For this reason, he proposes that the circle should be treated as a figural concept. Let us determine the correspondence of the points of the circle and then define it metrically or algebraically. All the points of the circle are equidistant (radius r) from point C (the center of the circle) and all the points equidistant from C are situated on the circle (see Figure 3). Algebraically, this is expressed by the following formula: $(x-a)^2+(y-b)^2=r^2$.

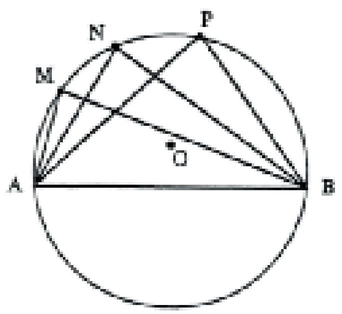

Figure 3: Comparing the angles directly and figurally according to Fischbein

Let us select two points, A and B, on the circle, and draw several angles over the tangent AB, with vertices M, N, and P (forming triangles AMB, ANB, and APB). It is difficult to compare the triangles directly and figurally (by relying on the image). The angles seem to be different. However, there is a theorem stating that all peripheral angles over one tangent, on the same side of a circle, are equal (Eudemus of Rhodes, 370 BCE – 300 BCE, philosopher, considered to be the first historian of science, believed that the theorem had been put forward by Thales of Miletus, 624 BC – 546 BC). Therefore, the three angles with the vertices M, N, and P are also equal. We are dealing here with figural concepts where all parts of the image (angles, sides, points, the circle, the arc) are simultaneously images and concepts (images are controlled by their definitions). However, it is obvious that in the process of reasoning the image cannot provide an answer to the posed question. The equality of the angles is determined by the theorem. Therefore, we can conclude that the image can lead to incorrect reasoning (drawing conclusions). All the vertices of the angles are placed on the same side of circle, with their vertices passing through the points on the circle (and have the same magnitude as an angle whose vertex lies on the circle). Fischbein believes that confronting figural impressions with formal constraints improves conceptual control and stimulates the merging of figural and conceptual constraints.

Logic and Image Should be Inseparable in GeometricReasoning

Logic and image should be inseparable in geometric reasoning, which the example of the locus problem clearly illustrates. Figural elements become a part of the process of drawing logical conclusions as if they themselves were the right example of the concept. Even if some discrepancy appears, it is usually due to the fact that the figure does not comply to the requirement, under the specific influence of the logic. Serbian well-known mathematician, Mihailo Petrović Alas (1868–1943) (Dejić, 2001, 2013), explored this problem area at the end of the 19th and the beginning of the 20th centuries. He proposed several situations that teachers should use when helping students to develop geometric reasoning when drawing geometrical figures.

Mihailo Petrović Alas - Example

Dejić observes that Petrović used several examples to demonstrate "how a small, and seemingly irrelevant incorrection, that is often impossible to spot while drawing, leads to the entirely wrong and incorrect geometrical conclusions, despite the fact that the given image is interpreted most correctly and accurately" (2001, p. 615–618). Dejić offers an interpretation of the history instruction of

Petrović (see Figure 4). Let us assume that AFB is an angle. Then we draw perpendiculars from the points A and B, located on the sides of this angle. Point C is their intersection. Then we draw a circle through the points A, B, and C. The points D and E are the intersections of the circle with the sides of the angle AFB.

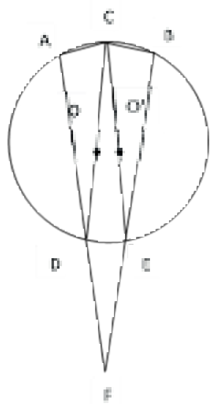

Figure 4: Incorrectly drawn image according to Mihailo Petrović Alas

As the angle DAC is a right angle, the line DC is one diameter of the circle, and O is the center of the circle. Because EBC is a right angle, EC is one diameter of the circle, and O' of the diameter is also the midpoint of the same circle. The conclusions drawn from the image are apparently correct. It turns out that this circle has several centers. Then where was the mistake made!? If we had drawn the image correctly, we would have noticed that the circle must pass through the point F, and not through the points D and E on the sides of the angle. In that case, the points D and E coincide with the point F, both parameters coincide with the diameter CF, and both centers fall into one point, in the real center of the circle. Dejić (2001) interprets Petrović's conclusions in the following manner: 1. by applying right reasoning, the incorrect result was obtained from the incorrectly drawn image and 2. to avoid serious mistakes, drawing should be as precise as possible. Precise drawing enables us to see the details that we otherwise would not pay attention to, such as the intersection of straight lines or arcs in the same point.

Fischbein's Commentaries on the Above-Mentioned Issues

According toFischbein (1993), students have to learn to mentally manipulate geometric objects and at the same time to apply operations with figures, logical correlations and operations. These mental activities involve the completion of the following tasks:1. drawing an image by unfolding a geometrical object and obtaining a folding (observed in reality or mentally represented), 2. identifying a geometrical object that can be created by imagining a bi-dimensional image, and 3. asking students which edges of a geometric object will match (fold) when the three-dimensional object is reconstructed. It is relatively easy to determine that Figure 5 represents the unfolding of a cube. It is more difficult to see (imagine) that the marked edges (sides 4 and 5 marked by arrows in Figure 5) also meet in the folded cube.

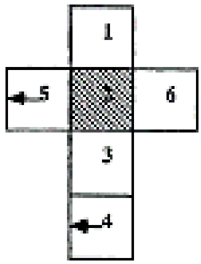

Figure 5: A possible shape of an unfolded cube according to Fischbein

Identifying an unfolded cube in Figure 6 would be even a more complex task. Similarly, it is not easy to realize that the marked edges (sides 1 and 6 marked by arrows in Figure 5) match in the reconstructed cube.

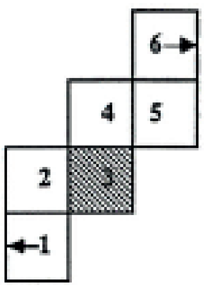

Figure 6: Another possible shape of an unfolded cube according to Fischbein

Fischbein further elaborates on the contribution of figural manipulations and logical operations. Images and concepts are generally considered to be separate categories of mental activities. When different types of mental transformations of three-dimensional objects are explored (such as rotation or folding and unfolding of figures), we realize that students deal with these operations as if they were of purely pictorial nature. But this is not true and it cannot be true because we are dealing with the sides of the cube (the examples provided above) with the edges of equal length, all the sides are squares, there are right angles, etc. This is all tacit knowledge which is implied in performing mental operations. Without such tacit conceptual control, the entire operation would be meaningless. Figural manipulation and mental logical operations contribute to the understanding of the concept and represent an "excellent opportunity for training the capacity of handling figural concepts in geometrical reasoning" (Fischbein, 1993, p. 157–159). According to Fischbein, the aim of such training would be the improvement of the following capacities: a constructive cooperation of figural and conceptual aspects of the activities for problem solving; coordination among as many figural-conceptual items as possible; organizing mental processes by using meaningful sub-units to reduce memory overload and anticipation; and, anticipating and connecting all the changes on the road to the final solution. Images and concepts interact in a cognitive activity (of children or adults) by cooperating in some situations or confronting one another in some other situations. However, the development of figural concepts is not a natural process. One of the main reasons why the topics related to geometry are so difficult in the school curricula is that figural concepts do not develop naturally, according to their ideal forms. For this reason, we recommend the formation of different types of didactical situations requiring a strict cooperation between an image and a concept that would result in their merging into one and unique mental object.

Didactical situations that would result in merging images and concepts into unique mental objects

These situations include: 1. tasks involving distances and, conversely 2. tasks where the patterns of the figures are naturally inclined not to follow conceptual constraints (resulting in tensions), or 3. tasks involving folding and reconstruction where the cooperation between logical requirements and figural concepts is very difficult. For this reason, our paper is aimed at identifying and finding such situations in the work with pre-service primary school teachers, such as the examples of correct and incorrect reasoning in geometry tasks provided in Appendices A–D.

TOPICS FOR FURTHER RESEARCH

The goal of our research will be to examine the correlation between figural (pictorial) and conceptual properties of geometric objects. The research objectives have been formulated as follows: 1. To investigate whether the figural (pictorial) structure dominates in the geometric reasoning of the pre-service primary school teachers over the formal conceptual constraints, and 2. To identify the situations in which the domination of one of the two properties occurs–the figural (pictorial) structure and the corresponding formal conceptual constraints.

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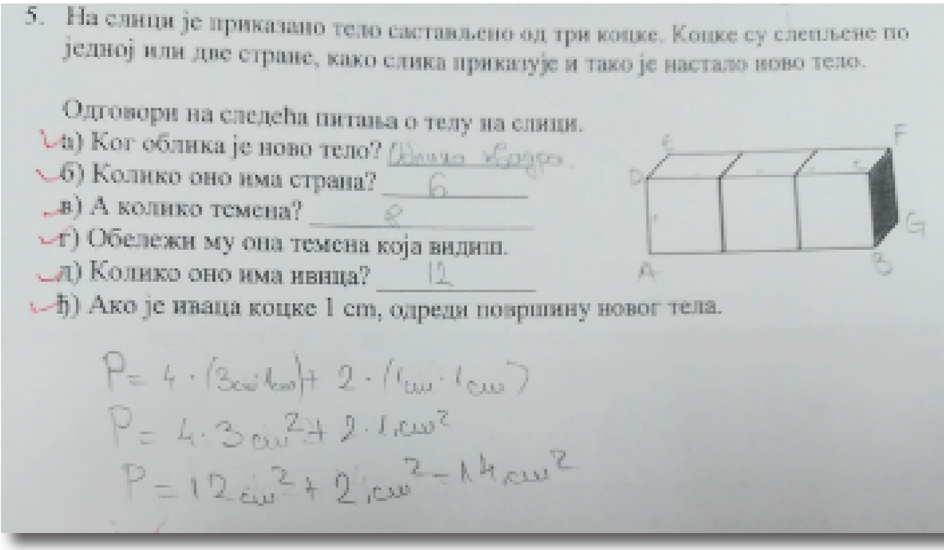
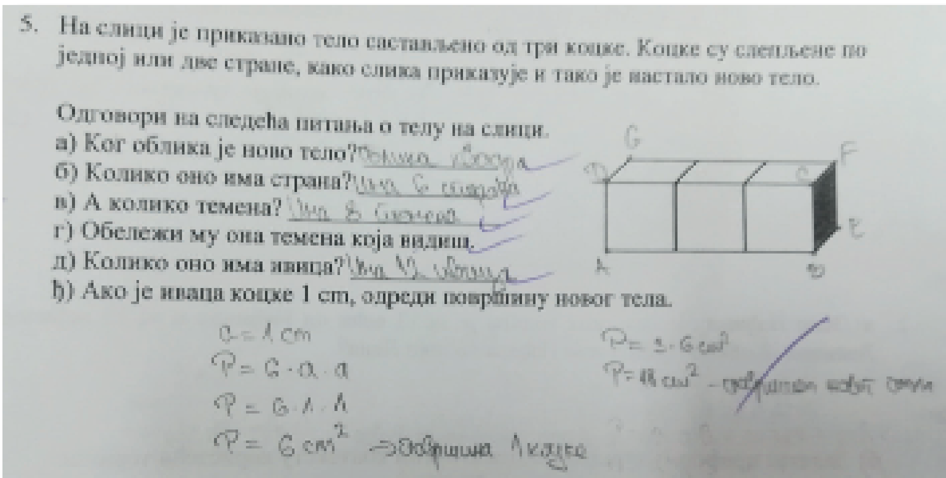
Appendix A



5. На слици је приказано тело састављено од три коцке. Коцке су слепљене по једној или две стране, како слика приказује и тако је настало ново тело.

Одговори на следећа питања о телу на слици.
а) Ког облика је ново тело? Облика квадра
б) Колико оно има страна? 6
в) А колико темена? 8
г) Обележи му она темена која видиш.
д) Колико оно има ивица? 12
ђ) Ако је ивица коцке 1 cm, одреди површину новог тела.

$P = 4 \cdot (3cm \cdot 1cm) + 2 \cdot (1cm \cdot 1cm)$
$P = 4 \cdot 3cm^2 + 2 \cdot 1cm^2$
$P = 12cm^2 + 2cm^2 = 14cm^2$



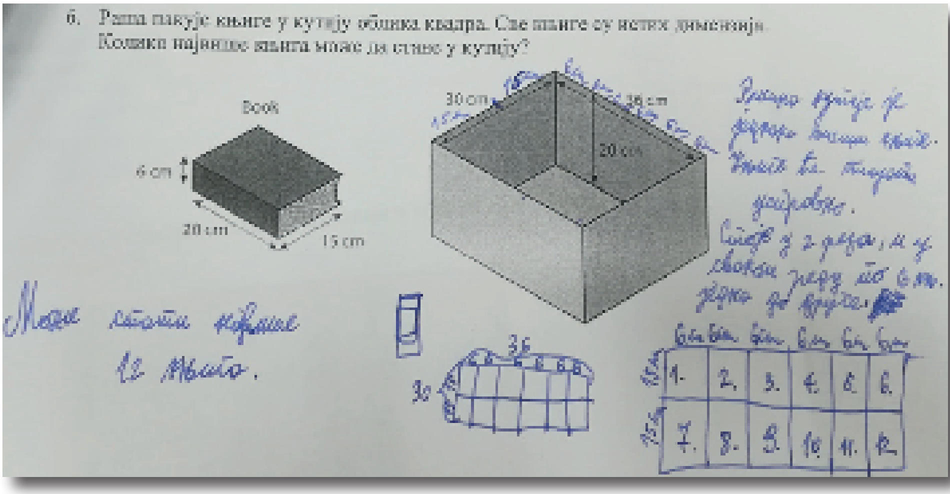
5. На слици је приказано тело састављено од три коцке. Коцке су слепљене по једној или две стране, како слика приказује и тако је настало ново тело.

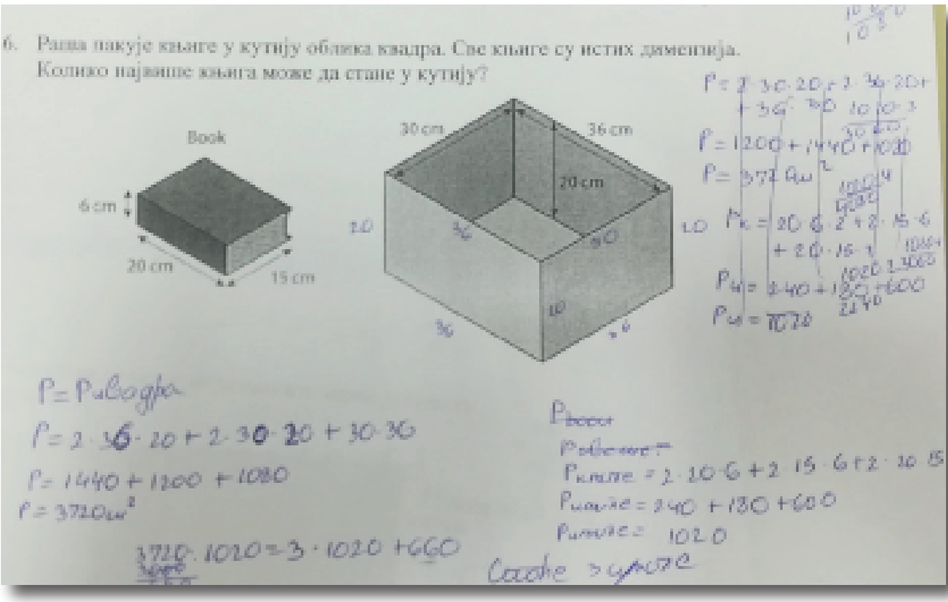
Одговори на следећа питања о телу на слици.
а) Ког облика је ново тело? Облика квадра
б) Колико оно има страна? Има 6 страна
в) А колико темена? Има 8 темена
г) Обележи му она темена која видиш.
д) Колико оно има ивица? Има 12 ивица
ђ) Ако је ивица коцке 1 cm, одреди површину новог тела.

$a = 1 cm$
$P = 6 \cdot a \cdot a$
$P = 6 \cdot 1 \cdot 1$
$P = 6 cm^2$ → површина 1 коцке

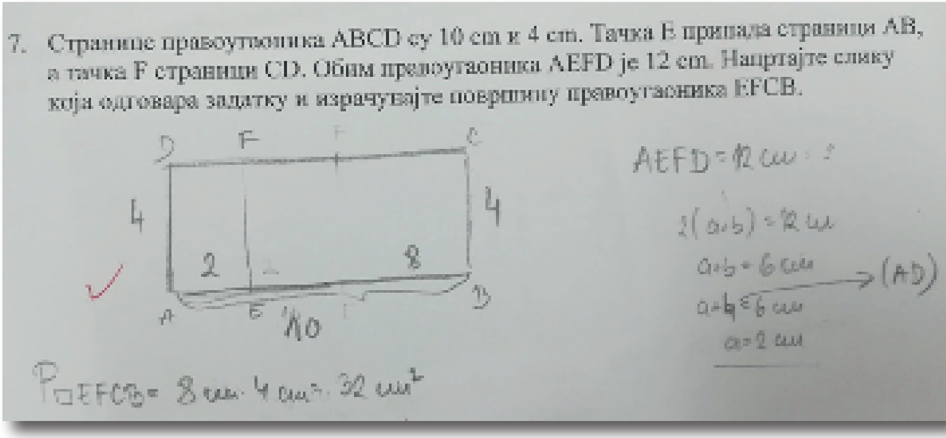
$P = 3 \cdot 6 cm^2$
$P = 18 cm^2$

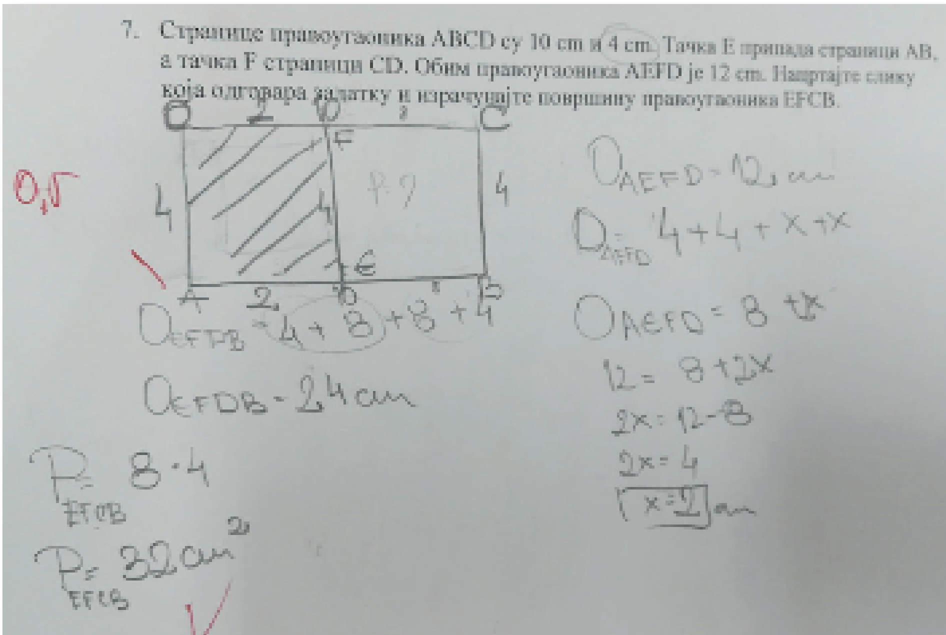
Appendix B


6. Раша пакује књиге у кутију облика квадра. Све књиге су истих димензија.
Колико највише књига може да стане у кутију?
Book
6 cm
20 cm
15 cm
30 cm
36 cm
20 cm
Може стати највише 12 књига.

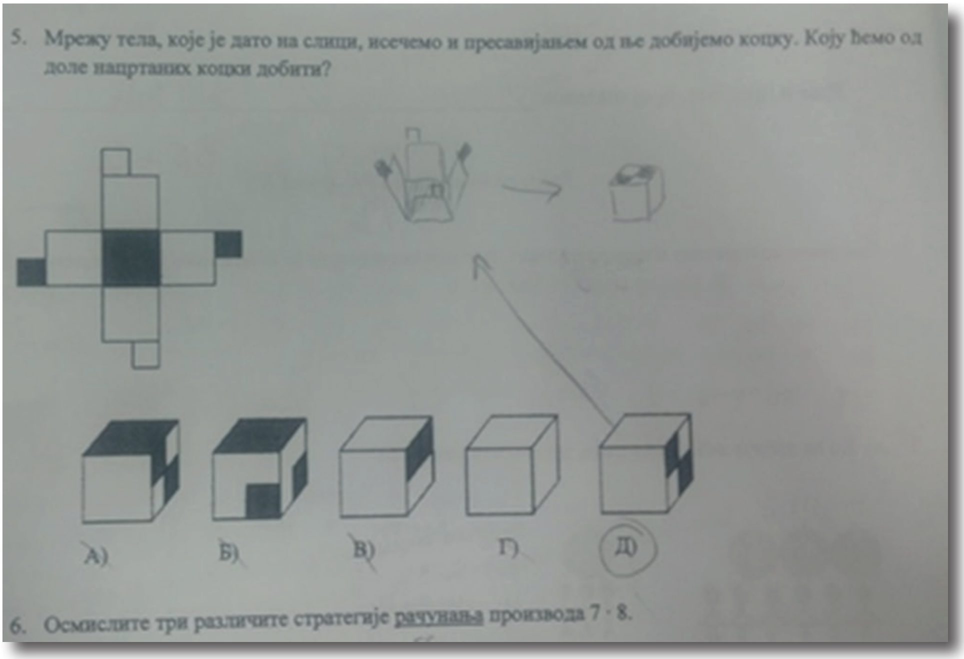

6. Раша пакује књиге у кутију облика квадра. Све књиге су истих димензија.
Колико највише књига може да стане у кутију?
Book
6 cm
20 cm
15 cm
30 cm
36 cm
20 cm

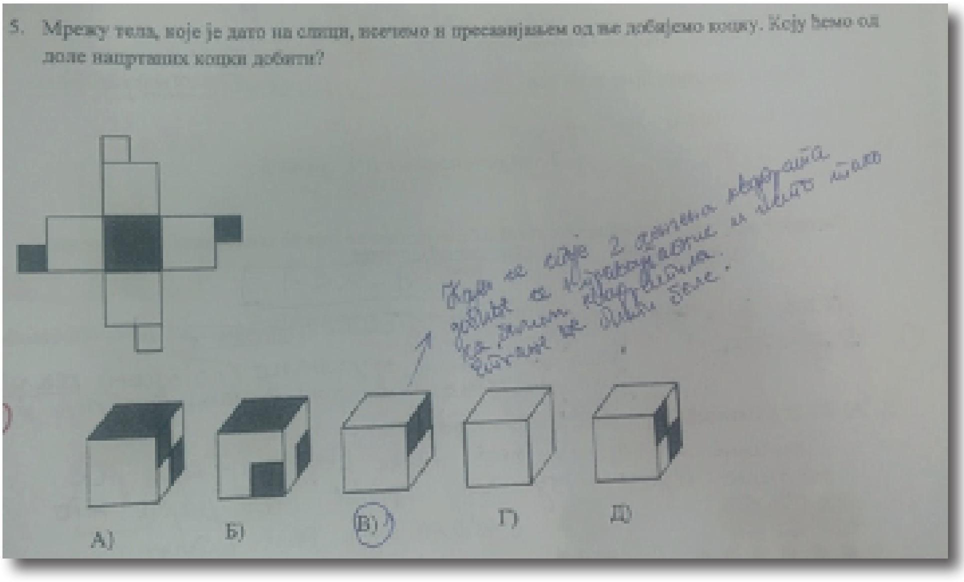
Appendix C


7. Странице правоугаоника ABCD су 10 cm и 4 cm. Тачка E припада страници AB, а тачка F страници CD. Обим правоугаоника AEFD је 12 cm. Нацртајте слику која одговара задатку и израчунајте површину правоугаоника EFCB.


7. Странице правоугаоника ABCD су 10 cm и 4 cm. Тачка E припада страници AB, а тачка F страници CD. Обим правоугаоника AEFD је 12 cm. Нацртајте слику која одговара задатку и израчунајте површину правоугаоника EFCB.

Appendix D


5. Мрежу тела, које је дато на слици, исечемо и пресавијањем од ње добијемо коцку. Коју ћемо од доле нацртаних коцки добити?
А)
Б)
В)
Г)
Д)
6. Осмислите три различите стратегије рачунања производа 7 · 8.


5. Мрежу тела, које је дато на слици, исечемо и пресавијањем од ње добијемо коцку. Коју ћемо од доле нацртаних коцки добити?
А)
Б)
В)
Г)
Д)

BINOMIAL COEFFICIENTS AND THEIR VISUALIZATION

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In this paper, the authors present some of the results achieved by mathematicians who belonged to different cultures and lived in different time periods, but were engaged in determining (formula for determining) binomial coefficients. Also, the authors present a geometric interpretation of the binomial coefficients of the binomial expansion $(a+b)^n$, *for* $n=1,2,3$ *and present an idea for the visualization of both binomial and polynomial coefficients that they plan to experimentally test in the upcoming period.*

Keywords: Binomial coefficients, binomial formula, polynomial coefficients, visualization, Pascal's triangle.

INTRODUCTION

What is the role of the history of mathematics in mathematical education? This issue attracts the attention of an increasing number of researchers and mathematics teachers. Studies of methodology of integrating history of mathematics into teaching and learning mathematics imply the following: consideration of philosophical, multicultural and interdisciplinary aspects that relate to this methodology; systematization of the spectrum of methods used for applying the given contents in the classroom, and the issues of practicability in relation to such implementation are discussed.

Binomial coefficients have been the subject of mathematical interest since Euclid's time. Binomial theorem is important for the development of algebra, but also for mathematical analysis. The first steps in determining binomial coefficients were made by Euclid in the 3rd century BC, when he gave geometric proof for the expansion of the binomial squares. The following diagram is referred to Chinese mathematician Chu-Shih-Cheih.

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1

1 1

1 2 1

1 3 3 1

. . . .

1 8. . . . . . . 8 1.

…

It is indicated that the diagram represents binomial coefficients. However, there is no evidence for such a claim (Goss, 2016). The German monk and mathematician Michael Stifel is also among the mathematicians who dealt with determining the binomial coefficients. He is credited for a diagram from the 16th century (Coolidge, 1949).

1

2

3 3

4 6

5 10 10

6 15 20

7 21 35 35

…

In the first column of the Stifel diagram, there are positive integers in the growing order. In each of the following columns, the given element is the sum of two numbers. The first addend is the number from the previous row, which is located directly above the requested element, while the second addend is the first number to the left of the first addend. Also, it can be seen that each odd line, starting from the third, has two equal numbers at the end.

However, for a diagram representing an infinite number of natural numbers in the form of a pyramid scheme, where each number represents the sum of the numbers above it (Figure 1), officially French mathematician Blaise Pascal (1632-1662) is credited.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Figure 1: Pascal's triangle

VISUALIZATION OF BINOMIAL COEFFICIENTS

Visualization of binomial coefficients can be achieved in several ways. One of the geometric interpretations of binomial coefficients for the binomial expansion $(a+b)^n$, for $n = 1, 2, 3$, involves the following.

- Splitting a line segment of the given length $a + b$ into two line segments with lengths a and b, $(n = 1)$.
- Dividing a square with side lengths $a + b$ into the square with side length a, two rectangles with side lengths a and b, and the square with side length b $(n = 2)$, i.e. $(a+b)^2 = a^2 + 2ab + b^2$.
- Dividing a cube with side lengths $a + b$ into the cube with side length a, three prisms with side lengths a, a and b, three prisms with side lengths a, b and b, and the cube with side length b, $(n = 3)$ i. e. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

An example of a scheme in the formation of higher-order concepts appears in a concrete example of the student Stephanie (Maher & Speiser, 1997). The girl determined the formula for calculating binomial coefficients by making connections between that problem with the problem of cubicle towers (problem she studied in elementary school). Maher and her colleague Al-Martino in 1989 began working with a small group of first grade students in elementary school, to encourage children to explore and explain their differences in opinions while solving specific problems. Stephanie was one of the students in this group and in these classes she was trying to find out how many different towers of the given height can be made from the two types of cubes which differed only by their colour. When she was in the eighth grade, she participated in the research (Maher & Speiser, 1997) where the authors worked on her understanding of mathematical problems, mostly algebraic. On this occasion, Stephanie calculated the binomial coefficient of the binomial expansion of $(a+b)^2$ and $(a+b)^3$ by connecting the task with the game of the towers.

We have used the given research to develop an idea for the visualization of binomial coefficients. Let's start with one concrete example. Suppose we want to determine the binomial coefficient with a^2b, in the binomial expansion of $(a+b)^3$ In order to bring this idea closer to students, we can assign a red cube to the first factor, i.e. a, and a yellow cube to b. Of course, all the cubes of the same colour are mutually equal. In this way, the binomial coefficient that correspond to a^2b represents the number of different towers made up of the two red and one yellow cubes. It is clear that there are three possibilities: red-red-yellow; red-yellow-red; yellow-red-red (Figure 2).

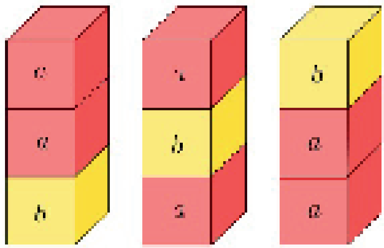

Figure 2: Visualization of the binomial coefficient that correspond to a^2b

Analogously, we can easily conclude that there are three different towers that correspond to the binomial coefficient of ab^2, which are created by the combination of one red and two yellow cubes. A simpler part in this case is the determination of binomial coefficients that corresponds to a^3 and b^3. It is obvious that the towers consisting of three red or three yellow cubes are unique. In this way, we determined all binomial coefficients in the binomial expansion of $(a+b)^3$ (Figure 3).

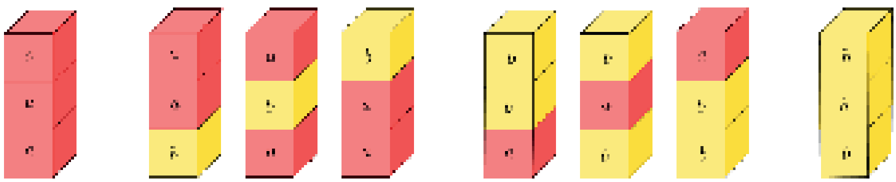

Figure 3: Visualization of the binomial expansion of $(a+b)^3$

In this way, we can determine the binomial coefficients for $a^{n-k}b^k$ of the binomial expansion of $(a+b)^n$ for a concrete value of the numbers n and k. The visualization of a given binomial coefficient is reduced to the determination of different towers consisting of the $n-k$ red cubes and k yellow cubes (whereby one red cube corresponds to one factor a and one yellow cube corresponds to one factor b of the product $a^{n-k}b^k$). In this way, students can experimentally be convinced that the binomial coefficient that corresponds to $a^{n-k}b^k$, of the binomial expansion of $(a+b)^n$ is equal to $\binom{n}{k}$.

Moreover, this scheme can be expanded and polynomial coefficients can also be presented in a similar way. For example, for determining the polynomial coefficients in the expansion of $(a+b+c)^4$, the polynomial coefficients will represent towers of height 4, i.e. towers consisting of four cubes. It is necessary, of course, to add cubes of another colour that would represent factor c, so we can add blue colour for this purpose. Thus, for determining the polynomial coefficient that corresponds to ab^2c, for example, we count different towers consisting of one red cube, two yellow cubes and one blue cube. It is easy to see that there are $\frac{4!}{2!}=12$ such towers. In general, the polynomial coefficient corresponding to $a^{k_1}b^{k_2}c^{k_3}$ in the expansion of $(a+b+c)^{k_1+k_2+k_3}$ is equal to $\frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!}$.

Encouraged by the results of Maher & Speiser (1997), we believe that the concept of representation of binomial coefficients, through towers made of cubes of appropriate colours, can be implemented in the teaching of mathematics. Our plan is to examine the influence of the visualization of binomial and polynomial coefficients through experimental research and to connect those coefficients with combinatorial elements.

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A HISTORICAL TIMELINE FOR THE ROLE OF MATHEMATICAL TASKS, STARTING FROM HOW TO SOLVE TASKS TO WHY WE SOLVE TASKS

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Mathematical tasks were a basic tool for strengthening mathematical knowledge in pre-Greek period. However, over time, tasks were replaced first by theorems and then by concepts. Therefore, historically there is a certain dynamics between a set of theorems and a set of tasks. Does this mean that with the change of the position of tasks through history their significance has been lost too?

Keywords: Mathematical tasks, historical timeline, didactic tool

Mathematical tasks are the main tool for achieving educational, practical and instructional aims of mathematics teaching. In order to achieve a long-term success, as well as a comprehensive adoption of the prescribed material, it is necessary to solve a significant number of tasks. Trying to emphasize the significance of mathematical tasks, it is sufficient to ask ourselves the following question: "Is mathematics teaching possible without mathematical tasks?" In the beginning, we will try to point out the role of mathematical tasks through history.

The first mathematical texts are the Egyptian papyruses: Rhind and Moscow. They contain formulations and solutions to specific problems of everyday practice. In these texts, rules are determined on solving tasks that can be reduced to solving linear and quadratic equations, as well as systems of equations. The Egyptians managed to calculate the approximate values of quadratic roots for some numbers. They learned the formula for the sum of the members of arithmetic progression, as well as the formula for the sum of squares of sequence of natural numbers. In Egypt, there is an improvement in approximating number pi, from the Babylonian value of 3 to 3.16. The formula for calculating the volume of a pyramid and the volume of a truncated pyramid with a square base are determined. Even then, they used the theorem now known as the Pythagorean and its reverse theorem. However, mathematics still did not exist as a science. In this period, the

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Egyptians and Babylonians successfully answered the question 'HOW?'; How it is done (calculated), without answering why it is done.

The answer to this very important question is obtained in Ancient Greece. In Ancient Greece, accurate was considered to be only that which could be reasonably proved. In Ancient Greece, systems of concepts and statements regarding these concepts were formed. The idea for deductive structuring of knowledge played the main role in these systems of concepts and statements, especially in geometry. The following steps occurred: grouping objects with similar characteristics, limiting the scope of the concept, creating definitions, looking into general characteristics…suddenly theorems were formed… With the occurrence of theorems, they became sufficient. Right then, ancient Greece began to change.

Mathematics in Ancient Greece was rich and comprehensive, more profound and extensive than any other intellectual activity created previously in Mesopotamia, Ancient Egypt, India and China.

The answer, or at least the attempt to answer the question how Western culture would have developed and looked like if ancient Greek heritage had not been included in its basis, would be considerably complicated and rather pointless. The influence that Ancient Greek philosophy, science and art had on the West **is impossible to estimate** – even today many consider 'classical' philosophy as a synonym to Ancient Greek philosophy, 'classical' drama as a synonym to Ancient Greek drama, 'classical' sculpting as a synonym to Ancient Greek sculpting. In fact, the achievements of those people relatively small in numbers and living in a relatively small area in the eastern Mediterranean, did not only spread towards the West, but also towards the East, following Alexander the Great's expedition, through the so-called Silk Road which connected Europe to China even during the Roman Empire. It is important to mention that the influence of Greek culture on India, Middle Asia, even the Japanese islands was spread in other ways as well.

When the Ancient Greeks assumed the leading role in science and culture, **mathematics** gained a new stimulus and direction for further development. While mathematical regularities of the Ancient Egyptians, Babylonians and Indus were obtained in empirical manner and adopted without proof, Ancient Greeks implemented the principle according to which mathematical regularities should not be accepted as true until they were proved. In this period, the inductive method was abandoned, and the deductive method developed, which had an immeasurable significance in further development of mathematics and other fields. Parallel with the development of the deductive method, the need arose for systemization of some mathematical disciplines, especially geometry. This entire developmental success of ancient mathematics lasted for several centuries, from the 6^{th} to the 2^{nd} centuries BC.

Before the Greeks, mathematics had mainly been in the hands of priests. They were the ‘knowledgeable’ ones who spread knowledge to whom they wanted and to the extent they wanted. They were the ‘masters’ in engineering and other activities where calculations were needed. It is important to mention here that they left their mark in mathematics.

Unlike them, the Greeks dispersed around the remote shores and islands developed as small states, and were ruled by wealthy citizens who obtained their wealth through trade. It is important to point out that the superiority of the priesthood was felt in lesser degree. In Greece, mathematics passed from the priests to the wealthy citizens. The first educated Greeks, known as the first philosophers, travelled through Egypt or Babylon. It was what Thales and Pythagoras did, as well as many others, including Plato. After these exhausting voyages, the future scientists could rest and contemplate. To some extent, their contemplation was conditioned by their status! On the one hand, they were independent from the ones who gave them knowledge, i.e. educated them and thus enabled science to develop independently from religion. On the other hand, they were not guided by the needs of their surroundings, but by their own wishes and abilities, seeking answers they were interested in, but which had no practical value, at least, for the immediate surroundings.

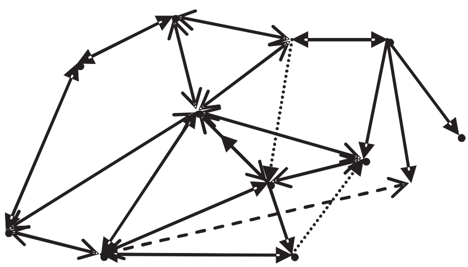
Among the first Greek mathematicians was Thales, a merchant from Miletus (624-547 BC). Thales was one of the Seven Wise Men of Greece, Phoenician by origin. He was the first to claim that the soul is immortal and predicted the solar eclipse of 28th May 585 BC. “Know thyself” and “Nothing in excess” are attributed to him. Travelling through Egypt, he familiarized himself with geometry and astronomy. In his old age, at his home he dedicated himself to science and his students. Thales is considered to be the founder of the so-called Ionian school that marked the beginning of a new age in history, the age of public schools. Thales was the first to adopt the principle of a ‘school available to anyone’. He invited students from all quarters of the educated world, and told them: ‘I will teach you everything I know myself’. The truth behind this statement was verbal, but nonetheless, Thales in defense of his claim often said: ‘It is like that’. Hence the famous dogmatic sentence ‘The teacher said so’ as an answer to the question ‘Why is that so?’ which, at that time, was often the only basis for proof. He was the first to form the proposition about the equality of right angles, the proposition for equality of the base angles of an isosceles triangle and its converse. He also knew some propositions about similar right triangles. He used denotations for congruent triangles and similar triangles. Thales managed to solve some practical geometric tasks, for example, calculating the height of the Cheops pyramid and the distance of ships from the shore. Thales determined the height of the Cheops pyramid by using a similarity with isosceles right triangles. He set up a stick on a flat sand surface and drew a circle around the base of the stick with a radius equal to the length of the stick. Thales waited for the moment when the shadow of the top of the stick was on the circle, that is, when the length of the visible part of the stick was equal to the length of the shadow. By measuring the stick's shadow he determined the height of the pyramid. Thales determined the distance of ships from the shore by using the similarity of some right triangles.

Starting from the first natural numbers and the simplest geometric figures, discussed as drawings of real-life objects, we can notice gradual accumulation of knowledge that, at a certain time incited leaps, presented by the birth of new methods, ideas, procedures, ..., firstly superficially, and then thoroughly with formed mathematical concepts and statements for their properties.

In ancient India, development of mathematical knowledge was connected with the creation of transferable symbols in arithmetic and algebra, the decimal number system, and the application of basic technical procedures in mathematics.

Mathematicians in the Arab world inherited what was created by the Greek and Indian mathematicians; they introduced some new points but did not manage to find an applicable symbiosis between the Indian technical procedures and Greek deduction. The main merit for the development of mathematics lies in the preservation of the ancient Greek achievements, the heritage of ancient Indian people and the capabilities that emerged from these in Europe. All this begins to be taught in Western Europe in the 13th century. Apart from these achievements on global level, mathematicians from Western Europe, with the help of the Arabs, are responsible for the occurrence of new array of knowledge and ideas regarding different specific phenomena.

The tools through which mathematical knowledge can be strengthened are: concepts, axioms, definitions, proofs of theorems, algorithms, tasks and their solutions. The concepts are the focal point. The axioms, definitions and theorems serve to connect different concepts. Therefore, the concepts, axioms, definitions, and theorems form a structure. This structure can be graphically presented in the following manner:



In the graph, the dots represent mathematical concepts. The bidirectional continuous arrows represent axioms. They connect the primary concepts. The unidirectional continuous arrows represent definitions. They connect each defined concept to the concept used in the definition. The arrows are directed from the defining concept towards the defined one. The bidirectional broke narrows represent theorems. They connect the defining concepts to the concepts we define or the primary concepts. Proofs can be modelled by 'constructing' bidirectional broken arrows representing theorems. In the graph, tasks, solutions to tasks and algorithms are not represented. Seeking the answer why is that so, we will look at the following:

Theorems in principle refer to the entire scope of the concept (contained in the definitions). Unlike theorems, tasks refer only to separate elements of the scopes of concepts or the real subsets of those scopes.

At that, these subsets are not formed as a scope to a certain concept. When this is done, the appropriate tasks become theorems. Therefore, historically there is certain dynamics between *the set of theorems and set of tasks.*

Tasks were a basic tool for strengthening mathematical knowledge in the pre-Greek period. Over time, tasks were replaced by theorems. Since then, tasks are a didactic tool wherever there is teaching.

Functions of tasks as a didactic tool

With the occurrence of theorems, the belief that tasks and the process of solving them are a suitable tool to adopt concepts and theorems, consolidate knowledge and develop skills to reason with axioms, definitions and theorems and thus determine the level of this knowledge, is gradually accepted. For these reasons, solving tasks is an important didactic tool in mathematics teaching.

The content of some non-mathematical tasks, as well as solving tasks as an activity with appropriate organization, can influence the formulation of important personal qualities such as: activity or passivity, thoughtfulness or negligence, increased interest or aversion to knowledge, sense for beauty, diligence or laziness, etc.

One very important activity for mathematics teachers is seeking answers to the questions "How", "Where", and "When" to solve tasks in the mathematics course.

Mathematical knowledge is used in different places, according to its role:

- to introduce new knowledge and skills (tasks as parts of a given theorem), to decrease the load of the proof of the theorem or complex procedures
- to detect new knowledge, i.e. solving tasks by introducing new knowledge

- to consolidate new knowledge and skills
- to detect skills for solving non-mathematical tasks using mathematical tasks
- to control, assess and consolidate knowledge and skills during oral examinations and while conducting different types of tests or quizzes
- to detect omissions in the students' knowledge and skills.

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FANGCHENG METHOD AS A TOOL FOR DEVELOPING PRE-ALGEBRA CONCEPTS IN PRIMARY GRADE STUDENTS

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In this paper we give an overview of the Ancient Chinese fangcheng method for solving systems of linear equations and discuss some possibilities of using this method in teaching primary grade students. The history of mathematics shows us that students very often form mathematical concepts in a way similar to the ways these concepts have been formed through the history of mankind. Taking into account these and the results of previous research, we believe that it is possible to use fangcheng method to develop some pre-algebra concepts in primary grade students.

Keywords: Fangcheng method, primary grade students, systems of linear equations

INTRODUCTION

The idea of integrating contents of the history of mathematics in mathematics teaching and learning is not new. There is extensive literature which offers a variety of reasons why and how to use history of mathematics in mathematics education. Wilson and Chauvot (2000) put forward four main benefits of using the history of mathematics in the classroom: (a) it sharpens problem-solving skills, (b) it lays foundation for better understanding of mathematics, (c) it helps students to make mathematical connections, and (d) it highlights interaction between mathematics and society. Liu (2003) gives five arguments for using the history of mathematics in school mathematics: (a) it helps to increase motivation and develop positive attitude toward learning, (b) the analysis of some past obstacles in the development of mathematics may explain the difficulties students face, (c) historical problems may help to develop students' mathematical thinking, (d) history reveals the humanistic aspect of mathematics knowledge, and (e) it gives teachers a guide for teaching. A number of studies points out that it is very important

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for future mathematics teachers to get to know the genesis of mathematical concepts and statements (Schubring et all., 2000; Dejić & Mihajlović, 2014). Guliker & Blom (2001) state that didactic skills of teachers are to be improved by reading old sources. It helps teachers to understand why certain concepts are difficult for their students. In the process of mathematics teaching and learning, the order of topics in the curriculum is determined mostly in accordance with historical development of some mathematical ideas (Katz, 1993). Furthermore, students very often form certain mathematical concepts in a similar way these concepts have been formed through the history of mankind: direct counting, measuring, observation of the real objects, etc. (Dejić & Mihajlović, 2014). Clearly, students do not go through the complete historical development (which sometimes lasts for centuries) in learning these mathematical concepts, but use shorter routes which can be facilitated by appropriate methodological transformation of mathematical contents.

In this paper we will discuss possibilities to introduce some pre-algebra concepts such as systems of linear equations to primary grade students by using Ancient Chinese fangcheng method. Since, there are not many empirical research studies that explore the benefits of using history of mathematics contents in mathematics classroom (Jankvist, 2009), we believe that our research will give a significant contribution to the field.

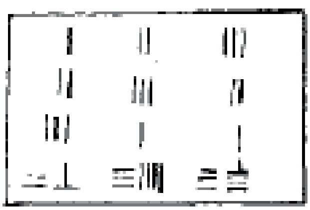
FANGCHENG METHOD

One of the most important mathematical works in China's long history is the *Jiuzhang Suanshu* or *Nine Chapters on the Art and Calculation* (Schwartz, 2008). The authors and date of the paper are not known, but it is believed that it was written shortly after 200 BC. The original version contained rules and algorithms, but no formal proofs or explanations. In the 263 AD Chinese mathematician Liu Hui provided written commentaries and justification for the techniques used. In the eighth chapter of the book, entitled *Fangcheng*, eighteen riddles/problems dealing with systems of linear equations and their solution techniques can be found. The word fangcheng can be translated as *matrices* or *rectangle arrays*. The problems/riddles were set up in a way similar to the way we would today formulate them in Linear algebra using n equations with n unknowns. They were based on practical applications to the real life situations. Chinese computation procedure was performed on a grid called counting board with a set of rods. Various rod configurations represented various numbers. The rod numerals were placed on the counting board in almost the same way we form the matrix for solving equations with multiple unknowns. The directions for solving problems required the setting up and manipulation of rod numeral rectangular arrays. The use of a counting board allowed Chinese to easily distinguish between different variables (Swetz, 1979). The analysis of these manipulations reveals that they used the same principle, which was developed centuries later in 1810 by Carl Friedrich Gauss, known as Gaussian elimination.

We will explain the counting board approach to the first problem given in Chapter 8, which involves a harvest of the three different grades of rice.

Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 dou. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 dou. Tell: how much [dou] does one bundle of each grade yield? (Schwartz, 2008)

The problem can be represented on the counting board (on the left) and by Arabic numerals (on the right) as given in Figure 1.



(I)

Left	Centre	Right
1	2	3
2	3	2
3	1	1
26	34	39

(II)

Figure 1. The counting board and Arabic numerals representation (Martzloff, 2006)

If we used the familiar algebraic notation, then this representation would be equivalent to the following set of equations

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

After carrying a series of multiplications and substractions with the appropriate columns, the initial "rectangular array" is reduced to a tabular form as illustrated in Figure 2.

1	6	3
2	9	2
3	3	1
26	102	39

1	0	3
2	5	2
3	1	1
26	24	39

3	0	3
6	5	2
9	1	1
78	24	39

0	0	3
4	5	2
8	1	1
39	24	39

0	0	3
20	5	2
40	1	1
195	24	39

0	0	3
0	5	2
36	1	1
99	24	39

Figure 2. Representation of solution on the counting board

We notice that the matrix of the system is reduced to a triangular form. By using a simple division, we can determine the value of one of the unknowns (z=36/99), and the other two unknowns can be found by successive substitutions. Martzloff (2006) points out that the fangcheng technique is "visibly nothing other than Gauss's method" (p. 254), although Gauss's work has little to do with the work of Chinese authors.

POSSIBILITES OF USING FANGCHENG METHOD TO DEVELOP PRE-ALGEBRA CONCEPTS OF PRIMARY GRADE STUDENTS

From the pedagogical point of view, history of mathematics has a potential to be used in school curricula. As Costa et al. (2015) point out, it can contribute to the development of the pre-algebra contents, such as systems of linear equations, unknowns and matrices. The same authors performed a case study about the use of a simplified form of fangcheng method for solving linear system of equations by a talented 10 year old student. Their study showed that it was possible to learn fangcheng method much earlier than the solving systems of linear equations is usually taught.

In Serbian primary schools students in lower grades learn how to solve some simple linear equations with one unknown, and how to model mathematics word problems into these equations. They learn how to solve systems of linear equations in 8th grade. However, in order to solve some complex word problems which require knowledge of some more advanced methods (such as systems of linear equations), primary school students (aged 9 and 10) use some mathematical models which allow them to transform these problems into more simple ones. Some of these models are: a line segment model (which corresponds to a bar model), a rectangle area model, and a false assumption method. For example, the rectangle area model is used in the case when one variable in the mathematics word problem can be expressed as the product of two other variables. The main idea is to present the relation of the three variables through the relation between the sides of the rectangle and its area. If one side of the rectangle is decreased, the other side should be increased in order for area to remain the same. By using a graphical representation in this model, the system of two linear equations with two unknowns is transformed into a process of solving the equation with one unknown.

Fangcheng method also relies on an adequate visual representation of numerals on the counting board and involves only basic calculations, such as multiplication and substraction. If we take into account these and the research by Costa et al. (2015), there is a reason to believe that primary grade students can understand and use the Fangcheng method to solve some mathematics word problems that involve systems of linear equations.

CONCLUSION

The history of mathematics offers teachers a number of guidelines how certain mathematical contents can and should be taught. However, there should be more empirical research in order to explore the effects of using history of mathematics in mathematics teaching. We plan to conduct an experimental study with 4th grade primary students (age 10) and explore the possibilities of using fangcheng method to solve mathematics word problems. The main goal of our study will be to investigate if primary grade students are able to learn and understand the fangcheng method.

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TANGRAM

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In this paper we present a brief history of tangram, a Chinese mathematical puzzle/game, which is more than three milleniums old. During the 19th century more than 6500 various problems (figures) related to tangram were solved, and this number is constantly growing. However, it is known that only thirteen convex shapes can be made.

Keywords: Tangram, Chinese math puzzle/game, solution (figure), convex figure.

ABOUT THE TANGRAM

Tangram is one of the oldest and most famous dissection puzzles (Lepšić & Ilić-Dreven, 1981).It is a mathematical puzzle[1], an interesting game which "engages" the brains and contributes to the development of creativity. According to a Chinese psychologist, although it was created for entertainment rather than for analysis, tangram is the oldest psychological test.

The very name *t a n g r a m* is a compound of two words: *tang* and *ram* (Loyd, 1903) and tangram in Chinese means "*seven tiles of wisdom* ".There is little known about the origins of Tangram, which means that the exact age of the game is unknown, but it is considered to be more than three thousand years old.

The legend says that a servant of a Chinese emperor, carrying a very valuable ceramic tile of square shape, tossed and fell and the tile broke into seven parts. Trying to put them in a square shape, the servant created various shapes of animals, people, and things.

1 A *mathematical puzzle* (ora "*brain teaser*") is *a* puzzle based on mathematical rules and the mathematical rules are mostly from the field of numerical theory or geometry. They are intended primarily for one player, who, based on the given conditions, should solve the task, i.e. the puzzle. Most mathematical puzzles cannot be solved without the knowledge of mathematics, as opposed to *mathematical games* where it is not necessary.

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Tangram game arrived in Europe (and other parts of the world) from China in the early 19th century.

T*angram* game gained great popularity in the first half of the 19th century, whereas the second wave of popularity occurred during the First World War.

Among the great tangram lovers were the famous American writer of crime novels Edgar Alan Po, and Napoleon, the French general who spent his days in captivity on the island of Saint Helena playing this game.

This mathematical puzzle consists of seven standard parts (Figure 1) of which the shapes of different objects are made. Traditionally, tangram tiles were made of stone, bone, clay, porcelain and similar materials, but today they are made of plastic, wood or hardboard.

The original tangram is square in shape, the side is one unit[2] (*jd*) long, cut into seven parts (see Figure 1). This way the following parts are obtained:

- two large isosceles right triangles with legs of length $\frac{\sqrt{2}}{2}$ (*jd*) and the hypotenuse of 1 (*jd*), and the area of $\frac{1}{4}$ (jd^2) – in image1 marked with *a*;
- one medium-sized isosceles right triangle with legs of length $\frac{1}{2}$ (*jd*) and the hypotenuse of $\frac{\sqrt{2}}{2}$ (*jd*) and the area of $\frac{1}{8}$ (jd^2) – in image 1 marked with *b*;
- two small isosceles right triangles with legs of length $\frac{\sqrt{2}}{2}$ (*jd*) and the hypotenuse of $\frac{1}{2}$ (*jd*), and the area of $\frac{1}{4}$ (jd^2) – in image 1 marked with *c*;
- one square with sides of length $\frac{\sqrt{2}}{4}$ (*jd*) and the area of $\frac{1}{8}$ (jd^2) – in image 1 marked with *d*; and
- one parallelogram with sides of length $\frac{\sqrt{2}}{4}$ (*jd*) and $\frac{1}{2}$ (*cm*), and the area of $\frac{1}{8}$ (jd^2) – in image 1 marked with *e*.

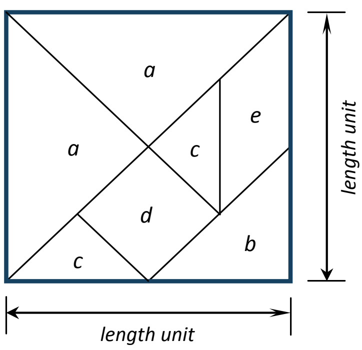

Figure 1.

2 Length units (*jd*) are: 1 *mm*, 1 *cm*, 1 *dm*, etc.

The game involves making predefined, different shapes from the cut pieces, according to one's own idea or, if there are multiple players participating in the game, according to the proposal of the game leader. In doing so, the following basic rules must be obeyed and respected:

1. *All the seven parts must always be used.*
2. *The parts are placed side by side and must not overlap.*
3. *Parts can be turned to the other side when needed.*

For those who have never tried to solve such mathematical puzzles/games this is an opportunity to do it now. And if you have solved such puzzles before, solve them again and return, at least briefly, to the carefree childhood.

In the following example we only give you some ideas and it is up to you to try to think of something new and creative.

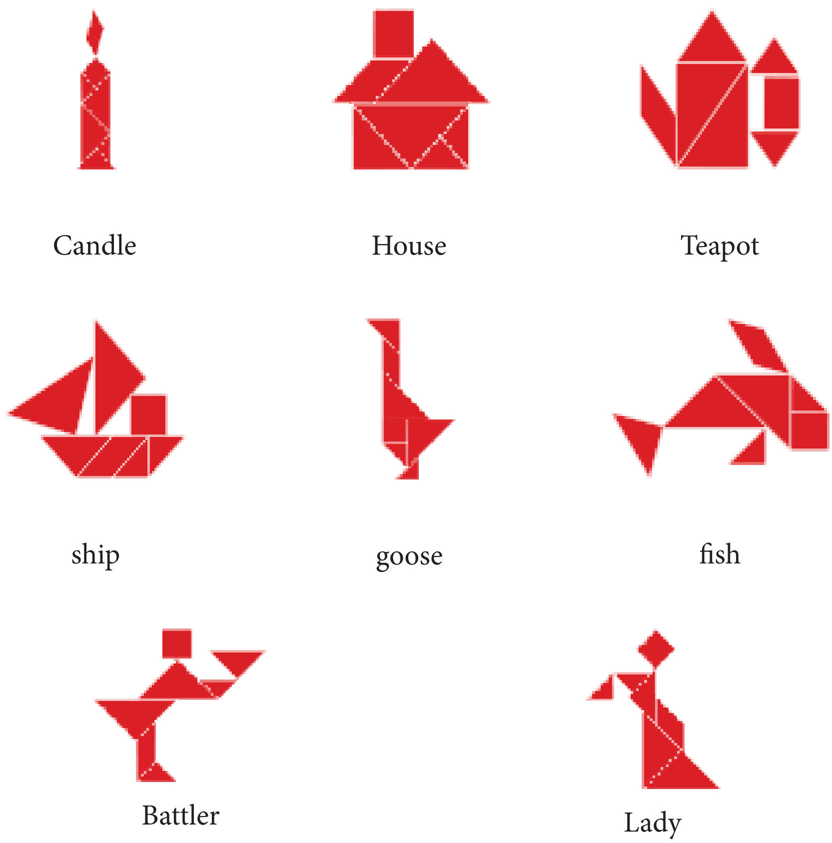

Figure 2. *E x a m p l e s* (taken from: https://hr.wikipwedia.org/wiki/ Datoteka:Tangram_203_Nevit.svg)

Solving the tangram requires full concentration and patience and is an excellent exercise for improving the ability to perceive and learn geometric shapes. It is recommended as an additional tool in the education of the youngest.

Only in the 19th century over 6500 different problems were solved and this number is constantly growing. However, only thirteen (13) convex shapes[3] can be made. The thirteen possibilities are shown in Figure 3.

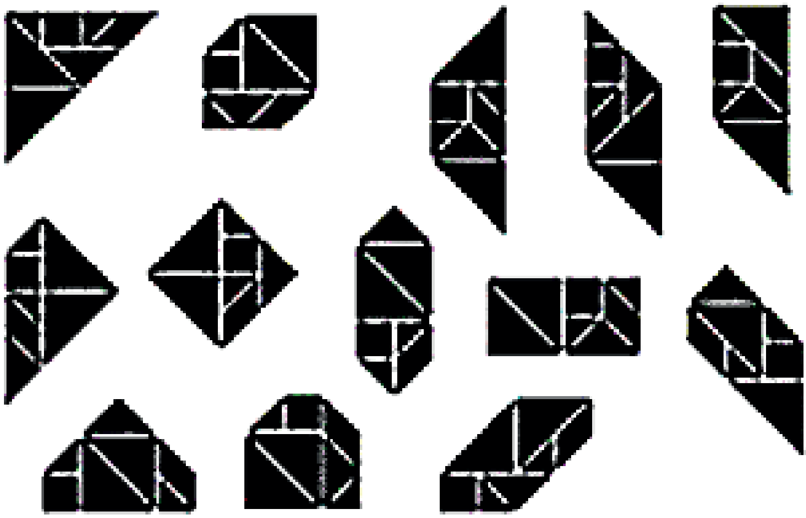

Figure 3.

The famous creators/inventors of such games were Henry Dudeney and Sam Loyd. Martin Gardner is also known for many new games he created / invented and released. Most of such assignments are published in the form of an interesting story or anecdote to avoid the dullness of the mathematical terminology.

A large number of puzzle newspapers and magazines, besides crossword puzzles, regularly publish mathematical puzzles for their readers' entertainment. Numerous mathematical puzzles can also be found and interactively played on the Internet.

THE USE OF THE TANGRAM IN TEACHING MATHEMATICS

In teaching mathematics, we should use games and mathematical puzzles as much as possible, because through mathematical games students develop skills,

3 A convex shape is a shape on which the length drawn between any two points on the perimeter lies completely inside the shape.

creativity, imagination and interest. They can be played for fun, but also for acquiring knowledge; they last relatively a short time and do not require special equipment; they are suitable for all ages and good knowledge of school mathematics is not necessary (Borković, 2017).

We can start playing the tangram game with students of the youngest age, i.e. with first and second graders. We may ask students to distinguish geometric shapes and try to design certain shapes by using only the shapes of one group, for example, all triangles.

During mathematics classes, third and fourth graders of elementary school can be given the assignment of drawing a tangram and cutting out its components. Then, they can continue with the stacking game where they can only use some parts and later be allowed to use all the remaining parts.

In higher grades of elementary school, students can be asked to determine the area of each individual shape which is part of the tangram if the area of the smallest shape is defined, i.e. if its area is P. Finally, based on the calculated areas of all individual shapes, they should be asked to define the area of the square as a function of the area P.

Student teachers doing the courses *Methodology of teaching mathematics* and *Teaching gifted students* can be given the assignment to create their own tangram by respecting the rules of making a tangram. Students can be asked to determine the area of each individual shape and the whole tangram as a function of the P area of the smallest shape. Further, students can be asked to determine the perimeter of each individual shape. In the end, they can try to create certain shapes or to create as many shapes as possible.

Here are some of the tasks that can be given to school or university students.

1. How many shapes are there in a tangram? (*seven*)
2. Which geometric shapes does the tangram consist of ? (*five isosceles right triangles, one square and one parallelogram*)
3. Are there any congruent, "equal" shapes among them? (*the two large triangles are congruent; the two small triangles are congruent as well; the remaining shapes are not congruent)*
4. If the area of the smallest triangles is P, compare the areas of all parts of the tangram. (*the medium-sized triangle is twice the size of the smallest triangle; therefore, its area is 2P; the largest triangle has the area twice larger than the medium-sized triangle; therefore, its area is 4P; the area of the square is twice larger than the area of the smallest triangles and its area is 2P; the area of parallelogram is twice larger than the area of the smallest triangle; therefore, its area is 2P* – methodical remark: everything mentioned should be clearly demonstrated by "covering" the larger shapes with smaller ones).

5. What is the area of the tangram? (*Now that we know the areas of all the parts of the tangram – see the previous task – the total area of the tangram is 16P which means that the area of the tangram is 16 times larger than the area of the smallest triangle*.)
6. Using two small triangles and a triangle of medium size, compose the following geometric shapes: (1) square; (2) rectangle; (3) triangle; (4) parallelogram /rhombus/; (5) trapezium.
7. What are the areas of such compound shapes /previous task /? (*all the shapes consist of the same parts, so it is easy to conclude that their areas are the same; this means that the area of all these shapes is the same and it is 4P; it can be concluded that d i f f e r e n t geometric shapes can have the s a m e areas.*)
8. Using the two small triangles, the medium-sized triangle, the square and the parallelogram, compose the following geometric shapes: (1) square; (2) rectangle; (3) triangle; (4) parallelogram; (5) trapezium.
9. Using all the seven parts of the tangram, compose the following geometric shapes: (1) square; (2) rectangle; (3) triangle; (4) parallelogram; (5) trapezium.
10. School (or university) students are already given compound shapes and their task is to find specific geometric shapes and/or some other shapes in them.

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CONTENTS OF THE HISTORY OF MATHEMATICS IN INITIAL MATHEMATICS TEACHING IN MONTENEGRO

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Being aware of the importance of the historical development of mathematics, this paper studies the representation of history of mathematics in primary mathematics teaching. We argue that the contents related to the history of mathematics should be taught to young learners as an important factor of encouraging children's interest in mathematics. The aim of the study is to examine the attitudes of primary teachers to the history of mathematics, its position in the curriculum of the lower grades of primary school, and teachers' readiness to teach these contents to the students. The survey was conducted on a sample of 130 participants. The results confirm the need for including history of mathematics contents in the curriculum of elementary grades, but they also point to a big failure of curriculum designers for omitting them from the compulsory teaching content.

Keywords: history of mathematics / student / teacher / math education / initial teaching of mathematics.

INTRODUCTION

If we take into account the specificity of mathematics education in primary grades, we can conclude that the first mathematical knowledge and mathematical concepts are related to the immediate environment of the child (Mićanović, 2014). This pathway of acquisition of mathematics knowledge coincides with the historical path of the development of mathematical concepts. Historical sources confirm that the mathematics had been practiced a long time before it was established as a science (Lazić & Lipkovski, 2013). The evidences can be found in the first traces of mathematical documents on the papyrus, clay tablets and on the walls of caves. The first mathematical texts from ancient Egypt and Mesopotamia were used for practical purposes, measurement of the land, construction of various buildings, etc.

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In the further process of helping children acquire mathematics knowledge, we gradually leave concrete and immediately noticeable objects in the environment and move to imaginary forms. These imaginary forms are called abstractions. Abstract mathematical concepts require intense concentration and deeper thinking, which stimulates the development of thinking skills. In order to encourage primary grade students to think and learn mathematical concepts, it is recommended to integrate the stories about the emergence and development of the learned concepts. History of mathematics that is used for this purpose has a strong motivational function (Dejić & Mihajlović, 2014). The development of logical mathematical thinking in lower grades of primary school depends upon the process of practice and intensity of understanding the content. Contents of the history of mathematics intensify the development of conative activities (Lawrence, 2014) as an important step in the development of mental abilities resulting in further improvement and development of mathematical skills, particularly logical thinking and reasoning. Although in lower grades of primary school there are actually no complex mathematical concepts, but the very first concepts are connected to the environment in which the child grows up, and they spontaneously introduce children to the world of mathematics and logical reasoning, keeping essential guidelines of mathematical approaches to understanding the functioning of the world in which we live, it is important from the very beginning to properly guide the pupils. Due to the absence of this, very often some problems occur, from dyscalculia which create serious difficulties in learning or understanding simple math (Tajar & Sharifi, 2011), to those related to difficulties in problem solving in mathematics (Tambychika & Meerah 2010).

METHODOLOGY OF RESEARCH

The aim of research. Research studies show that although different countries have similar approaches in mathematics education, there is still a contrast between different teaching cultures in many aspects (Kaiser & Vollstedt, 2007). The subject of our research was to determine the attitudes of primary teachers about applying the history of mathematics contents in mathematics teaching in primary grades. The aim of the research was to examine primary teachers' opinions about the history of mathematics, its position in mathematics curriculum of the lower grades of primary school, and the teachers' readiness to teach these contents to the students.

The main research hypothesis. We assume that teachers are familiar with the history of mathematical concepts which they teach, and that they include these contents in their lesson plans in the process of teaching mathematical concepts provided by the curriculum.

The research sample. The survey was conducted on a sample of 130 respondents. The sample is heterogeneous in terms of gender, educational level, and years of work experience.

The method of research. For the purpose of the study, data were collected through a questionnaire technique. The instrument was anonymous and contained both closed and open-ended items. Items were constructed according to the subject, aim and problem of research, and were related to: content knowledge about the history of mathematics, representation of historical contents in the primary mathematics curriculum, teachers' interest to teach historical content in mathematics classroom, usefulness of historical content in in achieving curriculum goals and objectives.

RESULTS OF RESEARCH

In order to answer the research questions, we surveyed teachers employed in primary schools in Podgorica (46 respondents), Niksic (43 respondents) and Budva (41 respondents). The description of the sample according to the years of work experience is given in Table 1.

Table 1: Sample according to the years of work experience

Sample groups	Work experience in teaching				**Total**
	From 0 to 10 years	From 10 to 20 years	From 20 to 30 years	Over 30 years	
Podgorica	13	19	9	5	**46**
Niksic	11	22	8	2	**43**
Budva	9	17	11	4	**41**
Total	**33**	**58**	**28**	**11**	**130**

Based on the data obtained, we can conclude that the majority of respondents belong to the category of 10 to 20 years of teaching experience (58 respondents), followed by the category from 0 to 10 years (33 respondents), followed by 20 to 30 years (28 subjects) and finally the category of more than 30 years of work experience (11 respondents).

One of the research questions was to investigate the extent to which teachers are familiar with the history of mathematics and its contents which could be applied in the primary grades. The answers according to the categories of years of work experience are given in the Table 2.

Table 2: The content of the history of mathematics by category of work experience

Sample groups	To what extent are you familiar with the contents of the history of mathematics?				**Total**
	very much	completely	partly	not at all	
From 0 to 10 years	11	17	5	0	33
From 10 to 20 years	23	28	7	0	58
From 20 to 30 years	9	12	7	0	28
Over 30 years	4	6	1	0	11
Total	**47**	**63**	**20**	**0**	**130**

The extent to which the respondents are acquainted with the contents of the history of mathematics are presented in Table 2. The majority of the participants in all categories of years of work experience stated that they were completely familiar with the contents of the history of mathematics (a total of 63 or 48.46% of the subjects). They are followed by those who were very familiar with the contents (47 or 36.15%) and partly familiar (20or 15.38%). There were no respondents who were not familiar with the contents of the history of mathematics. Therefore, based on the research sample, we may conclude that primary teachers are familiar with the contents of the history of mathematics, which is a good precondition for its application in teaching mathematics in the primary grades.

In the following research procedure we investigated the teachers' opinion about the extent to which the historical contents in teaching mathematics were present in the primary curriculum. The obtained results are presented in Table 3.

Table 3: Teachers' opinions about the extent historical contents in teaching mathematics in elementary grades are present in the primary curriculum

The group	The representation of historical contents in teaching mathematics in primary grades				**Total**
	very much	enough	partially	not at all	
First grade	0	0	5	125	130
Second grade	0	0	7	123	130
Third grade	0	0	7	123	130
Fourth grade	0	0	3	127	130
Fifth grade	0	0	5	125	130

Investigation of the representation of historical contents in teaching mathematics curriculum indicate that the attitudes of the respondents were homogeneous. Respondents believe that historical contents are not present in the mathematics curriculum for grades 1-5(95% of respondents). These findings are really worrying and indicate that the authors of the curriculum should dedicate more attention to the inclusion of the historical contents as an important basis for deeper studying of mathematical concepts at all age levels.

Table 4: Readiness of teachers to present historical contents in teaching mathematics in primary grades

The group	The interest of the teachers to present the historical content in mathematics				**Total**
	very much	completely	partly	not interested	
From 0 to 10 years	13	19	1	0	33
From 10 to 20 years	25	31	2	0	58
From 20 to 30 years	15	13	0	0	28
Over 30 years	5	6	0	0	11
Total	**58**	**69**	**3**	**0**	**130**

When it comes to the readiness of the participants for teaching historical contents in mathematics classrooms, research results indicate that a majority of teachers are completely (53.07%) or very much (44.61%) interested and ready for it, while a very small number of the respondents is partly (2.3%) ready for it. There were no teachers who showed lack of readiness to present historical contents together with teaching planned mathematical concepts in mathematics teaching.

Table 5: The usefulness of historical content for achieving the curriculum goals and objectives in mathematics teaching in elementary grades.

The group	The extent to which historical contents are useful for achieving the objectives and learning outcomes of the mathematics curriculum				Total
	Very much	completely	partly	not useful	
From 0 to 10 years	28	5	0	0	33
From 10 to 20 years	47	11	0	0	58
From 20 to 30 years	26	2	0	0	28
Over 30 years	8	3	0	0	11
Total	**109**	**21**	**0**	**0**	**130**

We wanted to examine the opinion of teachers about the usefulness of historical contents for understanding mathematical concepts, and the obtained results are encouraging. Most of the teachers replied that historical contents are very (83.84%) or completely (16:15%) useful for achieving the objectives and outcomes of mathematics curriculum in teaching mathematics in primary grades. We believe that this opinion stems from the experience of teachers and should be taken into account when developing the curriculum and planning lessons.

CONCLUSION

Mathematics education as a product of human civilization has its own history and profound influence on personality development. It was created out of practical needs, and for a long time mathematics has been reduced to the study of geometry and arithmetic, which presents the basis of the primary curriculum. Contents related to the history of the development of mathematical concepts as part of mathematics teaching are of great importance for the understanding of mathematical concepts as well as for students' motivation for mathematics, which makes them very useful in teaching mathematics. Since the focus of attention is a very important determinant in understanding the facts (Ognjenović, 2002), the contents of the history of mathematics should be part of the curriculum. The choice of teaching tasks is very important for the activity of students and teachers (Ni Zhou, Li & Li, 2014). A good understanding of teaching practice, well-developed communication between teachers and between pupils and teachers, and the selection of tasks, form the basis of successful learning (Olteanu, 2015). Pupils can be offered a variety of historical applications as illustrations (Hutchison, Beschorner & Schmidt-Crawford, 2012) necessary for understanding a concept.

The research results confirm the hypothesis *that teachers are acquainted with the history of mathematical concepts which they teach, that they plan and implement these contents in teaching mathematics together with teaching mathematical concepts.* The main problem in the implementation of these contents is the fact that primary mathematics curriculum does not include the contents of the history of mathematical concepts, and therefore teachers are not obliged to use them in teaching mathematics.

Presumption of successful teaching is that pupils are motivated and understand the contents being taught. Historical contents largely shed light on the need to study these concepts. Teachers play a central role in planning and implementation of instruction because they monitor the work and progress of students (Wixson &Valencia, 2011). It is, therefore, necessary to include and recommend contents from the history of mathematics in the curriculum as indispensable in teaching mathematics.

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HISTORY OF MATHEMATICS AS A FACTOR FOR INCREASING MOTIVATION IN THE MATHEMATICS CLASSROOM

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In this paper the authors consider the role of history of mathematics and some historical dataas a factor for encouraging interest and motivation of students to study mathematics. History of mathematics has a lot of questions, examples, and problems which can be adapted to cater for contemporary teaching and to become a direct support for content that needs to be processed, determined and deepened. Well-presented historical examples can have great importance in supporting mathematical and didactical meaning. The starting point in this paper is that "old" tasks preserved on the earliest memorials of mathematics, the anecdotes of great mathematicians and the ways of solving tasks can serve as a special kind of motivation for students' learning of mathematics because they introduce a special spirit and motive for solving such tasks and problems.

Keywords: History of mathematics, motivation, mathematics education, tasks.

In theoretical and empirical studies dealing with learning and teaching, more and more attention is paid to the development of learning through an educational process. Starting from the choice of content, its methodical transformation and optimal methods for their processing are increasingly being studied.

One of the main goals of the educational process is the development of students' opinions, so the greatest importance is attached to an understanding and thoughtful activation of students. The exceptional importance of basic education, especially mathematics, requires a large number of innovations in teaching that represent "original and specific changes that contribute to the rationalization, modernization, and efficiency of the teaching process" (Milijević, 2003, p. 11).

In 1962, a symposium dedicated to the initial teaching of mathematics was held in Budapest, organized by UNESCO, in accordance with the previous knowledge of psychology and didactics of mathematics (Piaget, Skemp, Galperin,

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Leontyev, Polya, Bruner, Dienes, Varga etc.), and attended by the most eminent scientists in this field.

It was then that the idea of mathematical education not just being possible for children, but also having a deep meaning and importance was established. Regardless of the abstract system of concepts in mathematics, it can be introduced already in the initial stages of teaching if, in accordance with the mental development of children, it starts from the game, the manipulation of things, that is, from the specific activities of the students, according to their interest.

Accordingly, basic mathematics education must be planned and taught so that, regardless of interest, the acquired knowledge and skills are permanently retained together with the achieved level of cognitive abilities. Modern mathematics should be an agent of communication and an "instrument" that is used in everyday life, to represent the connection between the child's experience of the world and mathematical structures, as well as to develop systematic and creative work. Thereby, the teaching of mathematics should be experienced as a process, or as a creative activity in which students actively participate.

The researchers of mathematical education (Poenkare, 1908; Polya, 1954; Freudenthal, 1980; Kooper, 1996) constantly point to the role and importance of intuitive, spontaneously acquired knowledge that includes elements of the history of mathematics in teaching as one of the necessary conditions for increasing motivation and interest of students in processing abstract mathematical content at all levels.

Cvetković asserts that "interest in mathematics and nurturing motivation to deal with it, can be developed by using data from the history of mathematics" (1980, p. 217). The same attitude is held by Dejić, who says that elements of mathematics contribute to "awakening" interest in mathematics (1997, p. 251).

The history of mathematics abounds with questions, examples, and problems that can be adapted to contemporary teaching and become direct support for content that needs to be processed, exercised, and deepened. Well-presented historical examples and tasks can be of great importance both in mathematical and methodical terms. Nevertheless, in order to allow higher quality materials to be better mastered, these tasks should encourage students to think, increase their interest and motivate them (Zlokas, p. 2017).

The first connection between mathematics education and the history of mathematics is about the history of mathematics education, especially modes and justifications. The researchers argue that reasons for the history of mathematics being so interesting can be divided into three themes. The first one sees the history of mathematics as encouraging multicultural approaches, giving students historical role-models, connecting the study of mathematics with human emotions and motivations (e.g., Swetz, 1995; Avital, 1995; Brown, 1993). The second one includes claims that the history of mathematics adds variety to teaching, decreases students' fear of mathematics, and so on. And the third theme claims that it is different when we speak about the influence that history has on teachers and a completely different one when we speak about influence on students.

There are two basic strategies for introducing the history of mathematics into the curriculum. Some researchers have given the name Strategy of addition to one of them, which is a passive strategy, and includes biographies of mathematicians. The second one, called Strategy of accommodation, includes using a historical development in one's explanation of a technique or organizing subject matter according to a historical scheme (Fried, 2001).

Also, researchers point to historical approaches to mathematics education, and argue that teachers can say that they need extra time for teaching history, and the thing is that teachers do not need extra time for it. They just should give students a historical problem, directly related to the topic they are teaching, or they can replace an ordinary classroom problem with one referring to the same material but having a historical context. A big issue of this story is relevance. Some researchers claim that the historical approach can work, in this view, since for every topic in the curriculum one can find a relevant historical problem, idea, or figure. But the downside of this continual measuring of relevance is that it turns the author of a historical approach in mathematics into a kind of editor of history, accepting what is relevant and throwing out what is not. "Mathematics teachers are committed to teaching modern mathematics and modern mathematical techniques naturally makes their relationship to the history of mathematics quite different from that of a historian of mathematics" (Fried, 2001, p. 398)

In the last part of the text, the author gives two directions toward resolving this difficulty: radical separation and radical accommodation. The first one represents the history of mathematics as a completely separate track compared to an ordinary course of instruction, and the second one is based on changing the study of mathematics into the study of mathematics text (Fried, 2001)

Some other research deals with involving the history of mathematics in mathematics education and a connection between history and pedagogy of mathematics. It also presents some meetings and related proceedings. One of them is European Summer University on the History and Epistemology in Mathematics Education, which is also one of the major activities in the HPM domain. They want to provide a school for working on a historical, epistemological, and cultural approach to mathematics and its teaching, with emphasis on actual implementation; they give an opportunity to mathematics teachers, educators, and researchers to share their teaching ideas and classroom experience related to a historical perspective in teaching; and motivate further collaboration along these lines among teachers of mathematics (Clarck, Cjeldsen, Schorcht, Tzanakis and Wang, 2016)

As something that can be interesting both for teachers and for students, is that we can use the history of mathematics to improve our methods and lesson plans. In that way, we can motivate students for their own research, so they can be introduced with more historical information that can be interesting for them, and which can be used in other subjects as well.

Some authors claim that integrating the history of mathematics with teaching and learning mathematics may force history to "serve" education, so it can be something like "Whig"- "…the present is the measure of the past. Hence, what one considers significant in history is precisely what leads to something deemed significant today" (Fried, 2001, p. 395).

A view in the history of mathematics shows that mathematics terms do not arise in abstract constructions for 'torturing' children, but for people's need to solve their own life problems and to master nature. In the past, people dealt with mathematics because there was a justified need for it, and that has led to the development of interest in this science, so it is clear that students will deal with mathematics if there is an interest in it. The first written facts, used as a basis on which we can surely speak of the level of development of mathematics, comes from the ancient civilizations of Babylon, Egypt, China, about 2,000 years BC. Thus, the first mathematics tasks were discovered on the slabs of the ancient Babylonians, on the papyrus of the ancient Egyptians and on the corners of the wood written by the ancient Chinese.

Students will be motivated by the fact that some tasks are very old, and we are still solving them today. Tasks like that "awake" more interest in solving. In the same way, students can be introduced with some of the tasks that are written in one of the oldest written mathematical findings (papyrus). One of the oldest findings of mathematical literacy is Moscow papyrus. It was written "during the reign of Pharaoh Senusret III or Amenemhet III, in the middle of the 19th century BC, and was discovered by the viceroy of the Russian Egyptology of the Golenishte (1892 or 1893) in Theba" (Živanović, 2012). The papyrus has 25 tasks with solutions. Some of them, which can be given to the children in lower grades of primary school and can be prepared for them are:

- *The number and its half give 9 when added. Find that number*.
- *Which even number added to its half and number 4 gives 10?* (Živanović, 2012, p. 129)

Ahmes papyrus too, which is dates back to 1650 BC, has ancient tasks which have been used even today. One of them is:

- *Size and its fourth totalled 15 units.*

The peculiarity of solving this task is that in solving it the ancient mathematician for the first time used the method of the false assumption.

The following task is from Ahmes calculation too.

In each one of seven houses live seven cats, every cat catches seven mice, every mouse eats seven barley ears, and every barley ear gives seven measures of barley. What numbers make this line and what is their sum? (According to Živanović, 2012).

This kind of a task brings into the mathematics classroom a special spirit and motive for solving it. To the pupils, the history and tasks that some ancient civilizations were solving, create a special mystery and they are eager to solve it. That inspires additional interest in studying mathematics.

Motivation and awaking new interest in mathematics can be achieved by tasks and anecdotes about great mathematicians. Pupils are especially interested in anecdotes about great mathematician Karl Friedrich Gauss whose teacher instructed his class to determine the sum of the first hundred natural numbers:

$1 + 2 + 3 + 4 + 5 + \dots + 96 + 97 + 98 + 99 + 100.$

The teacher expected the pupils to be engaged in solving the task and to remain calm for a long time. But, he was quickly surprised, when, after just a few moments, little Gauss gave the correct answer. So, by observing a series of given numbers, Gauss realized that the sums of pairs of natural numbers from different ends of the sequence are always of the same value:

$1 + 100 = 101$,
$2 + 99 = 101$,
$3 + 98 = 101$, etc. (Maričić, Špijunović, Cotič and Felda, 2017, p. 173).

Gauss came to the original idea for solving the task. Teacher today should use an example like this one to encourage students to be creative and express originality in the process of solving the tasks, and to be flexible in doing it, which can contribute to the development of divergent and creative thinking.

Motivation of awaking new interest in mathematics can be achieved by "interesting tasks", which contribute to the development of the ability to conclude, to think logically, to study conditions, causes, and consequences. One of these tasks is the task about a wolf, goat, and cabbage from the 8^{th}century.

The man must carry across the river a wolf, a goat, and cabbage. In the boat, beside the man there can only be a wolf, or a goat, or cabbage. But, if he left the wolf with the goat, the wolf would eat the goat; if he left the goat with cabbage, the goat would eat cabbage, and in presence of man "no one will eat anyone". The man managed to carry across would wolf, would goat, and cabbage. How did he do that?

There are many tasks from the history of mathematics which can be used in teaching mathematics with the aim of awaking interest of students in mathematics, and their motivation to research and study mathematical content. The main goal of introducing this content and the similar one is to increase participation of students in the classroom. Students who show interest in the content of study have a more expressive motive for achievement and through personal autonomy create an environment that suits his possibilities and interests, which is again a prerequisite for success.

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