THE RELATIONSHIP BETWEEN IMAGES AND CONCEPTS IN THE INITIAL GEOMETRY TEACHING

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Drawing upon Fischbein's theory of figural concepts, the starting point of the paper is the use and value of the history of geometry in mathematics education. Fischbein's theory is mainly based on a hypothesis that geometry deals with mental entities, the so-called geometrical figures, which simultaneously possess conceptual and figural properties. Fischbein called the geometrical figures figural concepts because of their nature. We shall analyze the internal tensions which, according to Fischbein, may appear in figural concepts because of their double nature, developmental aspects and didactical implications. In doing so, we have created a room for further research, the goal of which will be to examine the pre-service primary school teachers' geometric reasoning regarding the correlation between figural and conceptual properties of geometric objects in order to obtain a framework for creating didactic situations in which a figure and a concept would merge into a unique mental object.

Keywords: History instruction, geometric object, geometric reasoning, notion of figural concept, pre-service primary school teachers.

THEORETICAL BACKGROUND

Drawing upon the theoretical framework of development of students' geometric thinking (Dreyfus, 2014; Đokić & Zeljić, 2017), students' geometric reasoning (Đokić & Zeljić, 2017) and Fischbein's theory of figural concepts (Fischbein, 1993), the authors of the paper explore the use and value of the history of geometry in mathematics education (Đokić, 2017; Gulikers & Blom, 2001) as the starting point in their research. The main hypothesis of Fischbein's theory (1993) is that geometry deals with mental entities, the so-called geometrical figures that simultaneously possess conceptual and figural properties. Fischbein called the geometric figures *figural concepts* because of their nature. We shall analyze the internal tensions according to Fischbein which may appear in figural

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concepts because of their double nature, developmental aspects and didactical implications.

DEFINITION AND IMAGE

The relationship between a definition and an image is viewed differently in empirical sciences and mathematics (Fischbein, 1993). In empirical sciences, the definition is ultimately determined by the properties of the objects defined, whereas in mathematics the definition, created directly or by deduction, is the property of the corresponding objects. Therefore, the interpretation of the figural components of geometrical figures should entirely be subjected to formal constraints. This idea is not always clearly understood and students often tend to overlook it. The difficulty with manipulating figural objects lies in a tendency to neglect the definition due to figural constraints which are the main obstacle to students' geometrical reasoning.

The First Aspect of Clarity: Figural Concepts – Conflict Situations

Fischbein offers an example how to help students cope with this type of conflict situation. Students may not be able to draw correctly the altitude of the triangle ABC from vertex B (Figure 1), and they will draw BD instead, despite the fact that they know the definition of the altitude of the triangle.

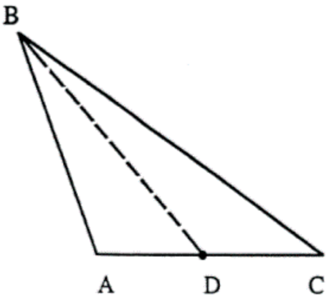

Figure 1: The problem with drawing triangle altitude according to Fischbein

The students should know the definition and solve the task by following it, and not according to what they think they see in the image. Many similar examples should be systematically used in the classroom, so that the domination of the definition over the image in interpretation of figural concepts becomes absolutely clear to all students. Fischbein provides another example. In this task, students have to compare the set of points in the segments AB and CD, and to resolve the conflict between two claims: that CD has more points than AB and the claim that two sets of points AB and CD are mutually equivalent (Figure 2).

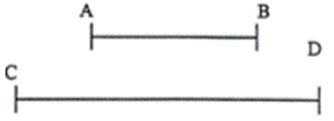

Figure 2: Comparing the sets of points according to Fischbein

The integration of conceptual and figural properties in a unique mental structure, with the predominance of conceptual constraints over the figural ones, according to Fischbein is not a natural process. Fischbein claims that "the processes of building figural concepts in the students' minds should not be considered a spontaneous effect of usual geometry courses" (1993, p. 154–156). This process requires teachers' continual, systematic, and committed involvement. Conflict situations should serve to teach students how to follow carefully the requirements set in definitions, which sometimes can run contrary to what the images seem to represent (or impose).

The Second Aspect of Clarity: Figural Concepts – the Concept of Locus

Fischbein asks a question whether a circle is only an image (i.e. a spatial representation). According to Fischbein, its existence and properties are imposed by an abstract, formal definition. For this reason, he proposes that the circle should be treated as a figural concept. Let us determine the correspondence of the points of the circle and then define it metrically or algebraically. All the points of the circle are equidistant (radius r) from point C (the center of the circle) and all the points equidistant from C are situated on the circle (see Figure 3). Algebraically, this is expressed by the following formula: $(x-a)^2+(y-b)^2=r^2$.

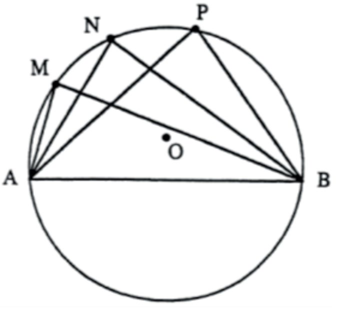

Figure 3: Comparing the angles directly and figurally according to Fischbein

Let us select two points, A and B, on the circle, and draw several angles over the tangent AB, with vertices M, N, and P (forming triangles AMB, ANB, and APB). It is difficult to compare the triangles directly and figurally (by relying on the image). The angles seem to be different. However, there is a theorem stating that all peripheral angles over one tangent, on the same side of a circle, are equal (Eudemus of Rhodes, 370 BCE – 300 BCE, philosopher, considered to be the first historian of science, believed that the theorem had been put forward by Thales of Miletus, 624 BC – 546 BC). Therefore, the three angles with the vertices M, N, and P are also equal. We are dealing here with figural concepts where all parts of the image (angles, sides, points, the circle, the arc) are simultaneously images and concepts (images are controlled by their definitions). However, it is obvious that in the process of reasoning the image cannot provide an answer to the posed question. The equality of the angles is determined by the theorem. Therefore, we can conclude that the image can lead to incorrect reasoning (drawing conclusions). All the vertices of the angles are placed on the same side of circle, with their vertices passing through the points on the circle (and have the same magnitude as an angle whose vertex lies on the circle). Fischbein believes that confronting figural impressions with formal constraints improves conceptual control and stimulates the merging of figural and conceptual constraints.

Logic and Image Should be Inseparable in GeometricReasoning

Logic and image should be inseparable in geometric reasoning, which the example of the locus problem clearly illustrates. Figural elements become a part of the process of drawing logical conclusions as if they themselves were the right example of the concept. Even if some discrepancy appears, it is usually due to the fact that the figure does not comply to the requirement, under the specific influence of the logic. Serbian well-known mathematician, Mihailo Petrović Alas (1868–1943) (Dejić, 2001, 2013), explored this problem area at the end of the 19th and the beginning of the 20th centuries. He proposed several situations that teachers should use when helping students to develop geometric reasoning when drawing geometrical figures.

Mihailo Petrović Alas - Example

Dejić observes that Petrović used several examples to demonstrate "how a small, and seemingly irrelevant incorrection, that is often impossible to spot while drawing, leads to the entirely wrong and incorrect geometrical conclusions, despite the fact that the given image is interpreted most correctly and accurately" (2001, p. 615–618). Dejić offers an interpretation of the history instruction of

Petrović (see Figure 4). Let us assume that AFB is an angle. Then we draw perpendiculars from the points A and B, located on the sides of this angle. Point C is their intersection. Then we draw a circle through the points A, B, and C. The points D and E are the intersections of the circle with the sides of the angle AFB.

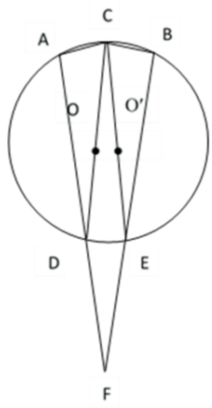

Figure 4: Incorrectly drawn image according to Mihailo Petrović Alas

As the angle DAC is a right angle, the line DC is one diameter of the circle, and O is the center of the circle. Because EBC is a right angle, EC is one diameter of the circle, and O’ of the diameter is also the midpoint of the same circle. The conclusions drawn from the image are apparently correct. It turns out that this circle has several centers. Then where was the mistake made!? If we had drawn the image correctly, we would have noticed that the circle must pass through the point F, and not through the points D and E on the sides of the angle. In that case, the points D and E coincide with the point F, both parameters coincide with the diameter CF, and both centers fall into one point, in the real center of the circle. Dejić (2001) interprets Petrović’s conclusions in the following manner: 1. by applying right reasoning, the incorrect result was obtained from the incorrectly drawn image and 2. to avoid serious mistakes, drawing should be as precise as possible. Precise drawing enables us to see the details that we otherwise would not pay attention to, such as the intersection of straight lines or arcs in the same point.

Fischbein’s Commentaries on the Above-Mentioned Issues

According toFischbein (1993), students have to learn to mentally manipulate geometric objects and at the same time to apply operations with figures, logical correlations and operations. These mental activities involve the completion of the following tasks:1. drawing an image by unfolding a geometrical object and obtaining a folding (observed in reality or mentally represented), 2. identifying a geometrical object that can be created by imagining a bi-dimensional image, and 3. asking students which edges of a geometric object will match (fold) when the three-dimensional object is reconstructed. It is relatively easy to determine that Figure 5 represents the unfolding of a cube. It is more difficult to see (imagine) that the marked edges (sides 4 and 5 marked by arrows in Figure 5) also meet in the folded cube.

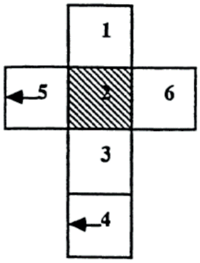

Figure 5: A possible shape of an unfolded cube according to Fischbein

Identifying an unfolded cube in Figure 6 would be even a more complex task. Similarly, it is not easy to realize that the marked edges (sides 1 and 6 marked by arrows in Figure 5) match in the reconstructed cube.

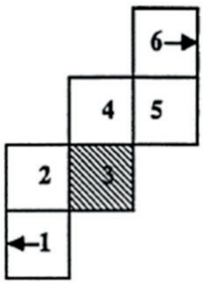

Figure 6: Another possible shape of an unfolded cube according to Fischbein

Fischbein further elaborates on the contribution of figural manipulations and logical operations. Images and concepts are generally considered to be separate categories of mental activities. When different types of mental transformations of three-dimensional objects are explored (such as rotation or folding and unfolding of figures), we realize that students deal with these operations as if they were of purely pictorial nature. But this is not true and it cannot be true because we are dealing with the sides of the cube (the examples provided above) with the edges of equal length, all the sides are squares, there are right angles, etc. This is all tacit knowledge which is implied in performing mental operations. Without such tacit conceptual control, the entire operation would be meaningless. Figural manipulation and mental logical operations contribute to the understanding of the concept and represent an "excellent opportunity for training the capacity of handling figural concepts in geometrical reasoning" (Fischbein, 1993, p. 157–159). According to Fischbein, the aim of such training would be the improvement of the following capacities: a constructive cooperation of figural and conceptual aspects of the activities for problem solving; coordination among as many figural-conceptual items as possible; organizing mental processes by using meaningful sub-units to reduce memory overload and anticipation; and, anticipating and connecting all the changes on the road to the final solution. Images and concepts interact in a cognitive activity (of children or adults) by cooperating in some situations or confronting one another in some other situations. However, the development of figural concepts is not a natural process. One of the main reasons why the topics related to geometry are so difficult in the school curricula is that figural concepts do not develop naturally, according to their ideal forms. For this reason, we recommend the formation of different types of didactical situations requiring a strict cooperation between an image and a concept that would result in their merging into one and unique mental object.

Didactical situations that would result in merging images and concepts into unique mental objects

These situations include: 1. tasks involving distances and, conversely 2. tasks where the patterns of the figures are naturally inclined not to follow conceptual constraints (resulting in tensions), or 3. tasks involving folding and reconstruction where the cooperation between logical requirements and figural concepts is very difficult. For this reason, our paper is aimed at identifying and finding such situations in the work with pre-service primary school teachers, such as the examples of correct and incorrect reasoning in geometry tasks provided in Appendices A–D.

TOPICS FOR FURTHER RESEARCH

The goal of our research will be to examine the correlation between figural (pictorial) and conceptual properties of geometric objects. The research objectives have been formulated as follows: 1. To investigate whether the figural (pictorial) structure dominates in the geometric reasoning of the pre-service primary school teachers over the formal conceptual constraints, and 2. To identify the situations in which the domination of one of the two properties occurs–the figural (pictorial) structure and the corresponding formal conceptual constraints.

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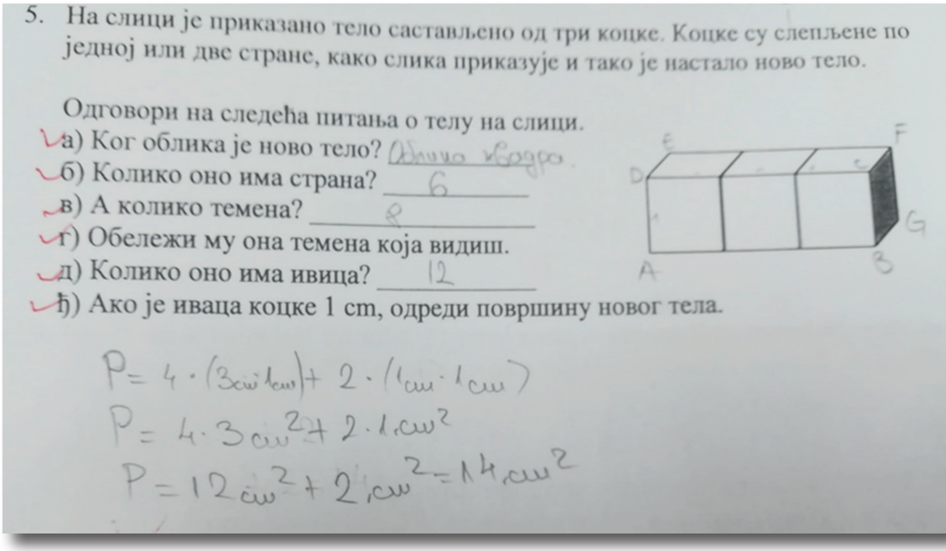
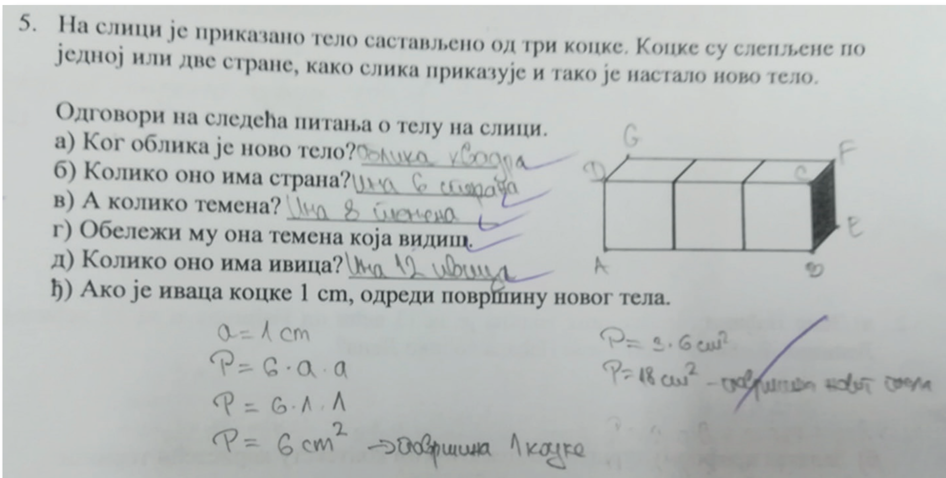
Appendix A



5. На слици је приказано тело састављено од три коцке. Коцке су слепљене по једној или две стране, како слика приказује и тако је настало ново тело.

Одговори на следећа питања о телу на слици.
а) Ког облика је ново тело? Облика квадра
б) Колико оно има страна? 6
в) А колико темена? 8
г) Обележи му она темена која видиш.
д) Колико оно има ивица? 12
ђ) Ако је ивица коцке 1 cm, одреди површину новог тела.

$P = 4 \cdot (3cm \cdot 1cm) + 2 \cdot (1cm \cdot 1cm)$
$P = 4 \cdot 3cm^2 + 2 \cdot 1cm^2$
$P = 12cm^2 + 2cm^2 = 14cm^2$



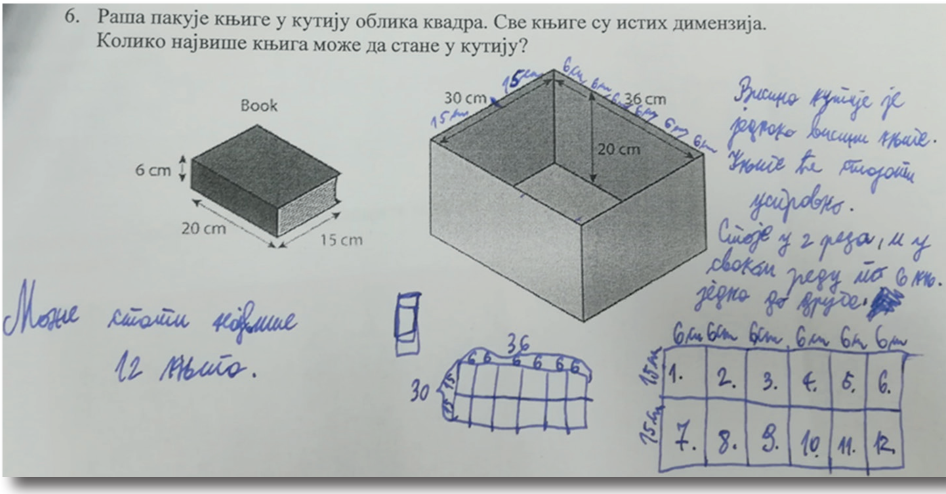
5. На слици је приказано тело састављено од три коцке. Коцке су слепљене по једној или две стране, како слика приказује и тако је настало ново тело.

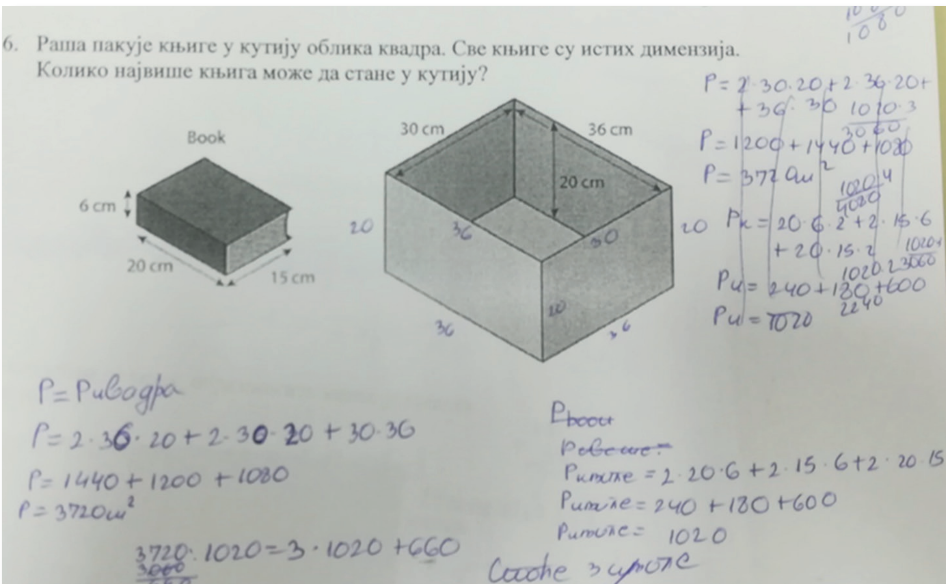
Одговори на следећа питања о телу на слици.
а) Ког облика је ново тело? Облика квадра
б) Колико оно има страна? Има 6 страна
в) А колико темена? Има 8 темена
г) Обележи му она темена која видиш.
д) Колико оно има ивица? Има 12 ивица
ђ) Ако је ивица коцке 1 cm, одреди површину новог тела.

$a = 1\,cm$
$P = 6 \cdot a \cdot a$
$P = 6 \cdot 1 \cdot 1$
$P = 6\,cm^2$ → површина 1 коцке

$P = 3 \cdot 6\,cm^2$
$P = 18\,cm^2$ – површина новог тела

Appendix B


6. Раша пакује књиге у кутију облика квадра. Све књиге су истих димензија.
Колико највише књига може да стане у кутију?
Book
6 cm
20 cm
15 cm
30 cm
36 cm
20 cm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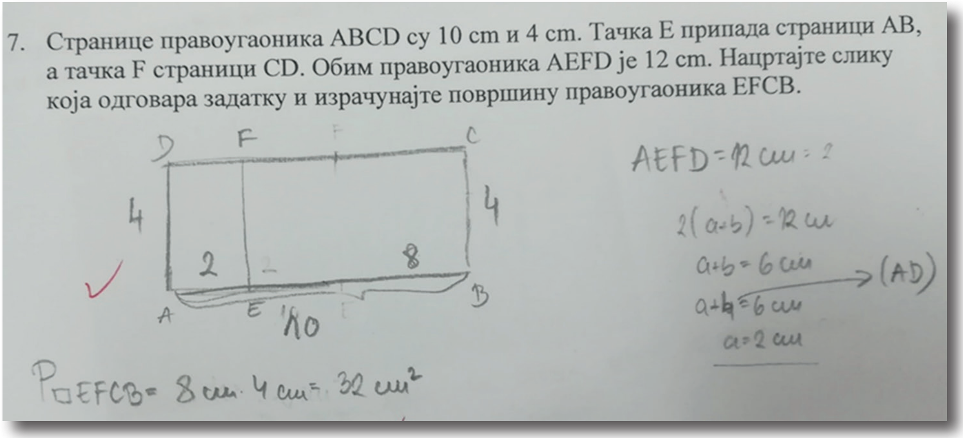
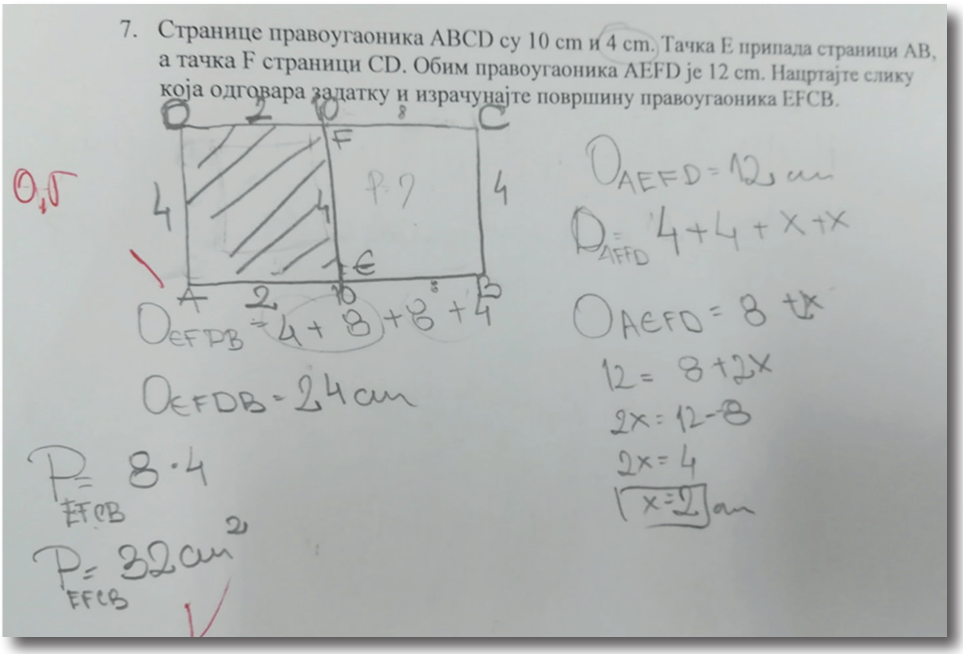
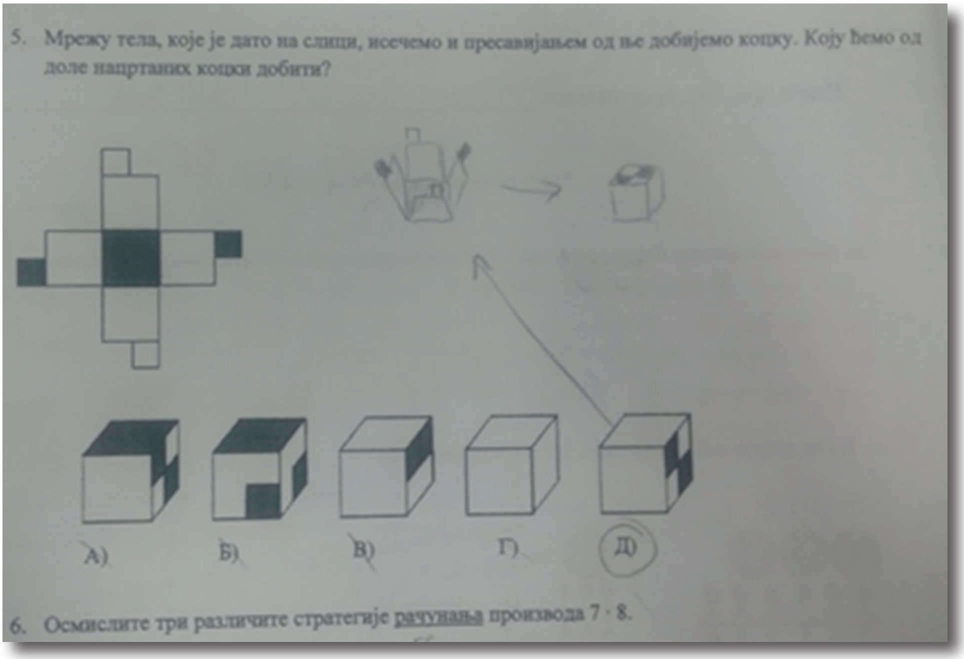
Колико највише књига може да стане у кутију?
Book
6 cm
20 cm
15 cm
30 cm
36 cm
20 cm

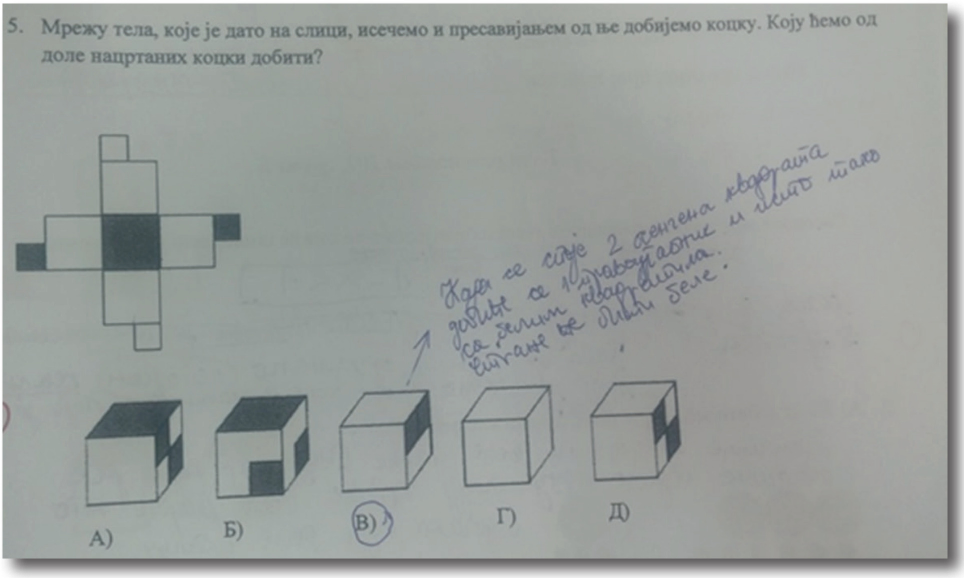
Appendix C


7. Странице правоугаоника ABCD су 10 cm и 4 cm. Тачка E припада страници AB, а тачка F страници CD. Обим правоугаоника AEFD је 12 cm. Нацртајте слику која одговара задатку и израчунајте површину правоугаоника EFCB.
AEFD = 12 cm : 2
2(a+b) = 12 cm
a+b = 6 cm
(AD)
a+4 = 6 cm
a = 2 cm
P□EFCB = 8 cm · 4 cm = 32 cm²


7. Странице правоугаоника ABCD су 10 cm и 4 cm. Тачка E припада страници AB, а тачка F страници CD. Обим правоугаоника AEFD је 12 cm. Нацртајте слику која одговара задатку и израчунајте површину правоугаоника EFCB.
0,5
O AEFD = 12 cm
O AEFD = 4+4+x+x
O AEFD = 8 + 2x
12 = 8 + 2x
2x = 12 − 8
2x = 4
x = 2 cm
O EFDB = 4 + 8 + 8 + 4
O EFDB = 24 cm
P EFCB = 8 · 4
P EFCB = 32 cm²

Appendix D


5. Мрежу тела, које је дато на слици, исечемо и пресавијањем од ње добијемо коцку. Коју ћемо од доле нацртаних коцки добити?
А)
Б)
В)
Г)
Д)
6. Осмислите три различите стратегије рачунања производа 7 · 8.


5. Мрежу тела, које је дато на слици, исечемо и пресавијањем од ње добијемо коцку. Коју ћемо од доле нацртаних коцки добити?
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