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INTEGRATING GEOMETRY AND ALGEBRA AS A WAY OF REIFICATION OF MATHEMATICAL CONCEPTS – HISTORICAL ASPECT

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The purpose of this paper is to illuminate the historical aspects of mathematics that could be used for the improvement of teaching practice. The idea that is already established in the literature is the existence of the similarities between historical development of mathematical concepts and development of concepts in the process of learning. Geometric algebra – the term which we use to describe content of the Euclid's Elements II is an important phase in the history of mathematics. On the other hand, geometry models in algebra are recognized as an important phase in the development of concepts in the learning process. The aim of this paper is to explore students' achievement in solving problems using geometry models. We also pose a question about the relationship between solving methods and students' age. If development of concepts follows their historical development, younger students should be oriented towards geometry visualizations and older to algebraic symbolism.

Keywords: Geometric algebra, visualization, history of mathematics

THE HISTORICAL DEVELOPMENT OF ALGEBRAIC CONCEPTS

Sfard (1995) uses the operational – structural model in the analyses of historical development of mathematics. She describes the development of algebra "as a hierarchy in which what is conceived operationally (i.e., as a computational process) on one level is reified into an abstract object and conceived structurally on a higher level" (Sfard, 1995, p. 16). She presents algebra learning as "a constant (but not necessarily conscious) attempt at turning computational procedures into mathematical objects, accompanied by a strenuous struggle for reification" (Sfard, 1995, p. 17). Reification is considered as a process of turning computational procedures into structural objects. This process requires the change of the

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way mathematical concept is comprehended – the processes on objects become objects themselves.

According to literature, the most frequently used phases in the development of algebra are Nesselmann's rhetoric, syncopate and symbolic algebra (according to Bagni, 2005; Kieran, 1992). However, numerous authors emphasize the Greek geometry as important phase in the historical development of algebra.

FROM EUCLID'S GEOMETRIC ALGEBRA TO THE INTEGRATION OF SYMBOLIC ALGEBRA AND GEOMETRY

The geometric algebra was based on interpretation of quantities expressed in the terms of line segments and figures. The aim of operating with these quantities is to find perimeters, areas or volumes of figures and shapes. By analyzing the historical development of algebra, Sfard highlights the procedural (operational) character of geometric algebra in the following way: "Greek geometry, with its "thinking embodied in, fused with graphic, diagrammatic representation" (Unguru, 1975, p. 76) was clearly at its structural stage, whereas algebra, preoccupied with verbally represented computational processes, could be conceived in no other way than operationally..... What algebra needed for further development was reification of its basic concepts. At this time, no better means were available to help in reification of the growingly complex computations than the palpable geometric objects." (Sfard, 1995, p. 23).

The content in Euclid's Elements was systematized in definitions, postulates, common notions and propositions. The books have geometry content, but today's interpretation of its proposition could be broader. The Second book is algebraic in nature. The Fifth book about proportions could be also understood as algebraic, with interpretation of algebraic problems in terms of geometry proportions. Books VII to IX are dedicated to theory of numbers - not the calculation procedures, but Pythagorean questions such as divisibility of whole numbers, sum of geometric progression, etc.

The procedure that could today be interpreted as algebraic, in Elements had geometric form. The expression $\sqrt{A}$ in Elements is the side of a square with area A, and expression $a{\cdot}b$ is area of rectangle with sides a and b. In Elements the request to determine one magnitude over others would be analogous with the construction with a compass and ruler. Kline argues that the roots of geometric algebra are related to Eudoxus (408-355 B.C.) who introduced the notion of a magnitude to denote entities like line segments, area, and volume (Kline, 1972, 48). Quantities are not assigned to these magnitudes (so Eudoxus ideas avoid irrational numbers) so this enables Antic Greeks to get generalized results. Kline's conclusion could be illustrated though the example of the fourth proposition of the Second book of Elements:

If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

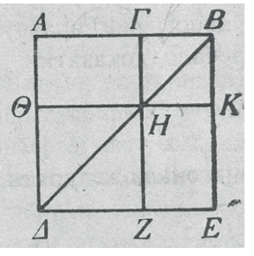

Figure 1: Euclid's Elements Book II Proposition 4.

Today, this proposition could be expressed as $(a + b)^2 = a^2 + b^2 + 2ab$ but in Elements Figure 1 illustrates its proof. The proof is in rhetorical form. We will illustrate the language in Elements through proof of one simpler proposition of Second book – Proposition 2 which states: If a straight line is cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole. Algebraic: if c = a + b then c^2 = ac + bc.

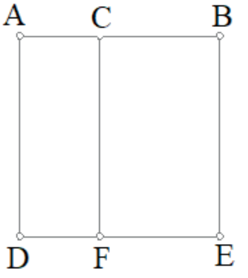

Figure 2: Euclid's Elements Book II Proposition 2.

Proof. For let the straight line AB be cut at random at the point C; I say that the rectangle contained by AB, BC together with the rectangle contained by BA, AC is equal to the square on AB. For let the square ADEB be described on AB [I. 46], and let CF be drawn through C parallel to either AD or BE. [I. 31] Then AE is equal to AF, CE. Now AE is the square on AB; AF is the rectangle contained by BA, AC, for it is contained by DA, AC, and AD is equal to AB; and CE is the rectangle AB, BC, for BE is equal to AB. Therefore, the rectangle BA, AC together with the rectangle AB, BC is equal to the square on AB. Therefore etc. Q. E. D. (Euclid, Elements, II. 2).

The development of symbolical algebra enabled mathematicians to express their ideas in a manner that is quicker and easier to communicate. Descartes (1596–1650) and Fermat (1601–1665) were the first to include symbolic algebra in geometry. They integrated geometry figures and transformations with calculation procedures. This was later called analytic geometry. Struik (1991) highlights that there are great differences between today's and Vieta's algebra. Vieta's algebra goes along with the Greek principle which treats product of length as area, and by which length only adds with length, area with area, and volume with volume. Descartes overcomes constrains that his predecessors had, which is recognized in Vieta's logistica speciosa, and considers x^3, x^2, $x \cdot y$ as line segments. Algebraic equations became relations between numbers. This was a huge step forward for mathematical abstractions.

GEOMETRIC ALGEBRA AS A SUPPORT FOR DEVELOPMENT OF ALGEBRAIC CONCEPTS

By opposing historical and ontogenetic development of algebraic thinking, Sfard (1995) emphases that difficulties that students have on different levels of learning mathematics could be close to difficulties that generations of mathematicians have experienced. She argues that there are reasons which refer to similarity of phylogenesis and ontogenesis of mathematical thinking. Katz and Barton (2007) highlight that historical motivation for developing algebra arises from the need for solving concrete, real and mathematical problems. They also argue that even if the most frequent starting points in algebra learning are generalized arithmetic and problem solving, geometry models could be used for the development of algebraic ideas. The examples are geometry concepts such as perimeter and area. Products of numbers could be represented as rectangles, and as an example, distributive property could be viewed as a statement about the rectangle from two different perspectives. These models are viewed as more concrete than real schematized object usually used in teaching practice.

It is hard to imagine that today's students are able to connect the above stated proposition (Figure 2)with distributive property $(a + b)c = ac + bc$ which is introduced in arithmetic through calculation procedures and justified in algebra through properties of algebraic structures. The rhetorical proof of the proposition gives an opportunity to students to use geometry visualizations and to construct a proof based on geometry statements. The question is if students are able to justify the proposition without the use of symbolic algebra.

Research has shown that today students and teachers tend to move to algebraic symbolism too quickly. This leads to absence of meaning of algebraic and geometric concepts and to an incomplete understanding (Kieran, 1992; Gravemeijer & Terwel, 2000). The use of mathematical formalism and absence of visualization in problem solving shows deficiencies in a methodical approach in earlier years of education.

DESCRIPTION OF THE RESEARCH

We have argued that historically, development of mathematical concepts relied on visualizations and geometry representations. In this paper we are asking if there is enough attention dedicated to visualization at the beginning of learning the basic mathematical concepts. Is the development of meaning based on geometry figures? The aim of this research is to investigate the students' achievement in solving problems by using geometry visualizations. We used propositions from the Second book of Elements as motivation for creating problems in the research. The problems are given in geometry context, but they could be solved algebraically. Historical development shows that those problems were the first solved using geometry visualizations and deduction and that the later development phase geometry transformations became proper algebraic transformations.

In this research we will analyze the responses of students of different ages. We want to answer the question if students of different ages use mathematical language which corresponds to different historical phases, i.e. if younger students tend to visually solve problems (like in ancient Greek phase) and older students tend to use algebraic transformations. The sample would be made from the students from 4^{th} and the 8^{th} grades of primary school. The test is created for the purpose of the research. The examples of tasks which will be used for the investigation of students' reification of concepts and the use of visualization is presented in Table 1.

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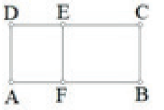
1. The side AB of rectangle ABCD is divided into two parts with length 3 cm and 5 cm. The side CD is divided also into two parts with length 2 cm and 3 cm. Calculate the area of rectangle ABCD.
2. The side AB of rectangle ABCD is divided into two parts with length *a* and *b*. The side CD is also divided into two parts with length *c* and *d*. Express the area of rectangle ABCD.
3. One side of rectangle is increased for 2 cm and the adjacent side for 3 cm. How will the area of rectangle be changed?

4. The rectangle ABCD is divided into square AFED and rectangle FBCE like in the picture. Express the area of rectangle ABCD if the area of square AFED is $a \cdot a$ and of rectangle FBCE $a \cdot b$.



5. The square with the side a is divided with two lines into two squares and two rectangles. Calculate the length of side a if one of the squares has area 25 cm^2, and one of the rectangles has area 60 cm^2.
6. The square with side a is divided with two lines into two squares and two rectangles. Express the length of side a if one of the squares has area b^2 and the other c^2. Also express the area of rectangles.

Table 1: Examples of tasks which will be used in research

Our hypothesis is that there is a tendency towards algebra from early grades. We want to investigate if students use visualization and visual problem solving, because it is an important part of the process of reification of mathematical concepts and procedures.

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