BINOMIAL COEFFICIENTS AND THEIR VISUALIZATION

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In this paper, the authors present some of the results achieved by mathematicians who belonged to different cultures and lived in different time periods, but were engaged in determining (formula for determining) binomial coefficients. Also, the authors present a geometric interpretation of the binomial coefficients of the binomial expansion $(a+b)^n$, *for* $n = 1, 2, 3$ *and present an idea for the visualization of both binomial and polynomial coefficients that they plan to experimentally test in the upcoming period.*

Keywords: Binomial coefficients, binomial formula, polynomial coefficients, visualization, Pascal's triangle.

INTRODUCTION

What is the role of the history of mathematics in mathematical education? This issue attracts the attention of an increasing number of researchers and mathematics teachers. Studies of methodology of integrating history of mathematics into teaching and learning mathematics imply the following: consideration of philosophical, multicultural and interdisciplinary aspects that relate to this methodology; systematization of the spectrum of methods used for applying the given contents in the classroom, and the issues of practicability in relation to such implementation are discussed.

Binomial coefficients have been the subject of mathematical interest since Euclid's time. Binomial theorem is important for the development of algebra, but also for mathematical analysis. The first steps in determining binomial coefficients were made by Euclid in the 3rd century BC, when he gave geometric proof for the expansion of the binomial squares. The following diagram is referred to Chinese mathematician Chu-Shih-Cheih.

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1

1 1

1 2 1

1 3 3 1

. . . .

1 8. . . . . . . 8 1.

…

It is indicated that the diagram represents binomial coefficients. However, there is no evidence for such a claim (Goss, 2016). The German monk and mathematician Michael Stifel is also among the mathematicians who dealt with determining the binomial coefficients. He is credited for a diagram from the 16th century (Coolidge, 1949).

1

2

3 3

4 6

5 10 10

6 15 20

7 21 35 35

…

In the first column of the Stifel diagram, there are positive integers in the growing order. In each of the following columns, the given element is the sum of two numbers. The first addend is the number from the previous row, which is located directly above the requested element, while the second addend is the first number to the left of the first addend. Also, it can be seen that each odd line, starting from the third, has two equal numbers at the end.

However, for a diagram representing an infinite number of natural numbers in the form of a pyramid scheme, where each number represents the sum of the numbers above it (Figure 1), officially French mathematician Blaise Pascal (1632-1662) is credited.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Figure 1: Pascal's triangle

VISUALIZATION OF BINOMIAL COEFFICIENTS

Visualization of binomial coefficients can be achieved in several ways. One of the geometric interpretations of binomial coefficients for the binomial expansion $(a+b)^n$, for $n = 1, 2, 3$, involves the following.

- Splitting a line segment of the given length $a + b$ into two line segments with lengths a and b, $(n = 1)$.
- Dividing a square with side lengths $a + b$ into the square with side length a, two rectangles with side lengths a and b, and the square with side length $b (n = 2)$, i.e. $(a+b)^2 = a^2 + 2ab + b^2$.
- Dividing a cube with side lengths $a + b$ into the cube with side length a, three prisms with side lengths a, a and b, three prisms with side lengths a, b and b, and the cube with side length b, $(n = 3)$ i. e. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

An example of a scheme in the formation of higher-order concepts appears in a concrete example of the student Stephanie (Maher & Speiser, 1997). The girl determined the formula for calculating binomial coefficients by making connections between that problem with the problem of cubicle towers (problem she studied in elementary school). Maher and her colleague Al-Martino in 1989 began working with a small group of first grade students in elementary school, to encourage children to explore and explain their differences in opinions while solving specific problems. Stephanie was one of the students in this group and in these classes she was trying to find out how many different towers of the given height can be made from the two types of cubes which differed only by their colour. When she was in the eighth grade, she participated in the research (Maher & Speiser, 1997) where the authors worked on her understanding of mathematical problems, mostly algebraic. On this occasion, Stephanie calculated the binomial coefficient of the binomial expansion of $(a+b)^2$ and $(a+b)^3$ by connecting the task with the game of the towers.

We have used the given research to develop an idea for the visualization of binomial coefficients. Let's start with one concrete example. Suppose we want to determine the binomial coefficient with a^2b, in the binomial expansion of $(a+b)^3$ In order to bring this idea closer to students, we can assign a red cube to the first factor, i.e. a, and a yellow cube to b. Of course, all the cubes of the same colour are mutually equal. In this way, the binomial coefficient that correspond to a^2b represents the number of different towers made up of the two red and one yellow cubes. It is clear that there are three possibilities: red-red-yellow; red-yellow-red; yellow-red-red (Figure 2).

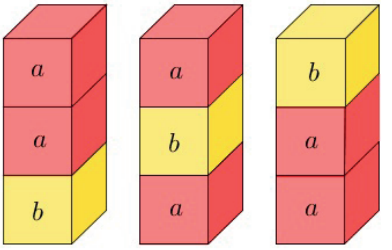

Figure 2: Visualization of the binomial coefficient that correspond to a^2b

Analogously, we can easily conclude that there are three different towers that correspond to the binomial coefficient of ab^2, which are created by the combination of one red and two yellow cubes. A simpler part in this case is the determination of binomial coefficients that corresponds to a^3 and b^3. It is obvious that the towers consisting of three red or three yellow cubes are unique. In this way, we determined all binomial coefficients in the binomial expansion of $(a+b)^3$ (Figure 3).

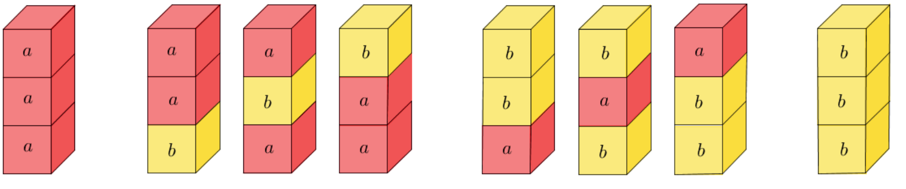

Figure 3: Visualization of the binomial expansion of $(a+b)^3$

In this way, we can determine the binomial coefficients for $a^{n-k}b^k$ of the binomial expansion of $(a+b)^n$ for a concrete value of the numbers n and k. The visualization of a given binomial coefficient is reduced to the determination of different towers consisting of the $n-k$ red cubes and k yellow cubes (whereby one red cube corresponds to one factor a and one yellow cube corresponds to one factor b of the product $a^{n-k}b^k$). In this way, students can experimentally be convinced that the binomial coefficient that corresponds to $a^{n-k}b^k$, of the binomial expansion of $(a+b)^n$ is equal to $\binom{n}{k}$.

Moreover, this scheme can be expanded and polynomial coefficients can also be presented in a similar way. For example, for determining the polynomial coefficients in the expansion of $(a+b+c)^4$, the polynomial coefficients will represent towers of height 4, i.e. towers consisting of four cubes. It is necessary, of course, to add cubes of another colour that would represent factor c, so we can add blue colour for this purpose. Thus, for determining the polynomial coefficient that corresponds to ab^2c, for example, we count different towers consisting of one red cube, two yellow cubes and one blue cube. It is easy to see that there are $\frac{4!}{2!}=12$ such towers. In general, the polynomial coefficient corresponding to $a^{k_1}b^{k_2}c^{k_3}$ in the expansion of $(a+b+c)^{k_1+k_2+k_3}$ is equal to $\frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!}$.

Encouraged by the results of Maher & Speiser (1997), we believe that the concept of representation of binomial coefficients, through towers made of cubes of appropriate colours, can be implemented in the teaching of mathematics. Our plan is to examine the influence of the visualization of binomial and polynomial coefficients through experimental research and to connect those coefficients with combinatorial elements.

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